Dynamics and Stability AE3-914

Sample problem—Week 4

Satellite in a four-dimensional universe

Statement

Consider a satellite with mass m orbiting about a planet in a universe with four spatial dimensions. In such a universe the gravitational field can be shown to be governed by the potential function

$$V(r) = -\frac{km}{r^2}.$$

Assume the orbit of the satellite to be planar and investigate the stability of the steady motion of the satellite in this orbit.

Model

The orbit of the satellite is assumed to be planar. This means that a system of polar coordinates (r, θ) is sufficient to describe the motion.

Lagrangian function

The kinetic energy is given by the expression

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{1}$$

in a system decribed in polar coordinates. This can be found through a change of coordinates to x and y or by observing the radial and the transversal components of the velocity vector.

Considering the potential given in the statement, the Lagrangian reads

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{km}{r^2} \end{aligned} \tag{2}$$

Ignorable coordinates

Variable θ is not found explicitly in the Lagrangian. Consequently, θ is an ignorable coordinate and an integral of motion can be found as

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = C_\theta. \tag{3}$$

Routhian function

The ignorable variable θ can be removed from the formulation by means of the Routhian function

$$R = C_\theta\dot{\theta} - L. \tag{4}$$

Solving $\dot{\theta}$ from (3) one gets

$$\dot{\theta} = \frac{C_\theta}{mr^2}, \tag{5}$$

which can be substituted into (4) to get

$$\begin{aligned} R &= \frac{C_\theta^2}{mr^2} - \frac{1}{2}m\left(\dot{r}^2 + r^2\frac{C_\theta^2}{m^2r^4}\right) - \frac{km}{r^2} \\ &= -\frac{1}{2}m\dot{r}^2 + \frac{C_\theta^2}{2mr^2} - \frac{km}{r^2} \end{aligned} \tag{6}$$

Steady motion

The conditions for steady motion are found by imposing that

$$\dot{r} = \ddot{r} = 0, \tag{7}$$

which, when making use of the Routhian formalism, are equivalent to

$$\frac{\partial R}{\partial r} = 0. \tag{8}$$

In our case this is elaborated as

$$\frac{\partial R}{\partial r} = -\frac{C_\theta^2}{mr^3} + \frac{2km}{r^3} = 0, \tag{9}$$

which can be reworked to the condition

$$\dot{\theta}^2 = \frac{2k}{r^4}. \tag{10}$$

Notice that this expression is obtained by substituting the integral of motion (3) into the condition (9).

Effective potential

Inspection of the Routhian (6) reveals a quadratic term in $\dot{r}$ and some more terms, including the potential V. This is an indication that the behaviour of the system is governed by an effective potential rather than by the gravitational potential exclusively. The proper (and safe) way of obtaining the expression for the effective potential goes through deriving the expression of the Jacobi energy integral from the Routhian,

$$\begin{aligned} h &= R - \frac{\partial R}{\partial \dot{r}}\dot{r} \\ &= \frac{1}{2}m\dot{r}^2 + \frac{C_\theta^2}{2mr^2} - \frac{km}{r^2}. \end{aligned} \tag{11}$$

where the effective potential

$$V_{\text{eff}} = \frac{C_\theta^2}{2mr^2} - \frac{km}{r^2} \tag{12}$$

is recognised.

Stability of the steady motion

The steady motion is stable if the effective potential attains a strict minimum. This is studied on the concavity of the effective potential through differentiation. The first derivative of V_{eff} is

$$\frac{\partial V_{\text{eff}}}{\partial r} = -\frac{C_\theta^2}{mr^3} + \frac{2km}{r^3}. \tag{13}$$

Compare this result with (9) and realise that the steady motion condition is literally equivalent to the condition

$$\frac{\partial V_{\text{eff}}}{\partial r} = 0, \tag{14}$$

which is analogous to the usual equilibrium condition in statics. The concavity of the effective potential in the steady motion is studied through the second derivative

$$\frac{\partial^2 V_{\text{eff}}}{\partial r^2} = \frac{3C_\theta^2}{mr^4} - \frac{6km}{r^4}. \tag{15}$$

Substitution of the integral of motion (3) and the condition for steady motion (10) now (certainly *not* before differentiating V_{eff}) renders

$$\frac{\partial^2 V_{\text{eff}}}{\partial r^2} = 0. \tag{16}$$

In view of this result it cannot be decided yet whether the effective potential attains a minimum for the steady motion. According to the standard procedures of differential calculus, a function exhibiting zero first and second derivatives attains a minimum if the first non-zero derivative is of even order and positive. In this case it is immediate to verify that

$$\frac{\partial^n V_{\text{eff}}}{\partial r^n} = \frac{(-1)^n (n+1)!}{r^{n+2}} \left(\frac{C_\theta^2}{2m} - km \right), \tag{17}$$

which is identically zero for any order n when conditions (3) and (10) are substituted. This means that the effective potential is actually constant in the neighbourhood of the steady position and that, therefore, no force is taking the satellite back to this position when small deviations take place. A closer examination of V_{eff} in equation (12) reveals that if the steady motion conditions are substituted, it vanishes for any value of r, so actually no forces at all are driving the motion of the satellite.

In view of the above considerations it can be concluded that the steady motion of the satellite under the considered potential is *unstable.*

Remark

The fact that in a universe with four spatial dimensions the gravitational potential would have the form given in the statement and therefore the orbits of celestial bodies would be unstable as shown, is used by Stephen Hawking in his recent work *The Universe in a Nutshell* to state that if our three-dimensional universe were a subset of a four-dimensional universe, the Solar System would not exist.