System Validation

Mohammad Mousavi

7. Modal mu-Calculus

Modal μ-Calculus

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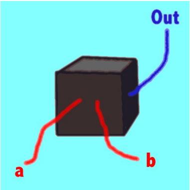
TU/Eindhoven

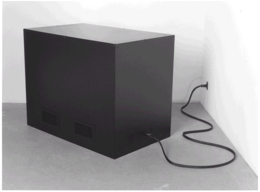
System Validation, 2012-2013
TU Delft

Outline

Specification using Temporal logic

- Fix observable events (interactions with external world)





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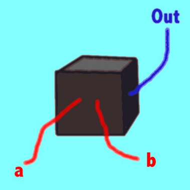

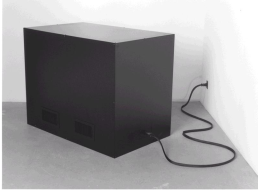

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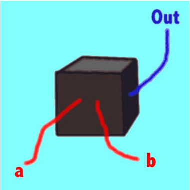

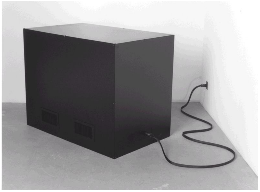

- Describe temporal properties using these
- Verify the correctness of the properties with respect to some labeled transition system

A scientist interacts with its environment

- coffee for taking coffee in
- coin for producing a coin
- pub for producing a publication
- ...

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Properties of interest

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- the scientist is willing to drink both coffee and tea now

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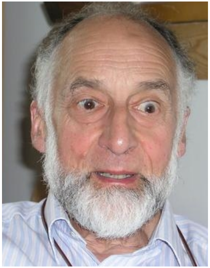
Properties of interest

- the scientist is not willing to drink coffee now
- the scientist is willing to drink both coffee and tea now
- she always produces a publication after drinking coffee

Outline

Hennessy-Milner logic

- Introduced by Hennessy and Milner in 1985



Matthew Hennessy and Robin Milner. Algebraic laws for nondeterminism and concurrency. Journal of the ACM, 32(1):137-161, 1985.

Syntax of Hennesy-Milner logic

for $a \in Act$

$$F ::= true \mid false \mid \neg F \mid F \wedge F \mid F \vee F \mid \langle a \rangle F \mid [a]F$$

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- There is a minimal subset

Syntax of Hennesy-Milner logic

For $A = \{a_1, \cdots, a_n\} \subseteq Act$ with $n \geq 1$

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In the book, ‘true’ is also used for ‘Act’.

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$$\langle \mathrm{coffee}\rangle \mathit{true} \wedge \neg\langle \mathrm{tea}\rangle \mathit{true}$$

Examples

- the scientist will always produce a publication immediately after having drunk two coffees in a row

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$$[\text{coffee}][\text{coffee}](\langle \text{pub} \rangle \mathit{true} \wedge [\mathit{Act} \setminus \{\text{pub}\}]\mathit{false})$$

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- the process is deadlocked
- the process can execute some action
- *a* must happen next
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- *F* holds after one step

$$[Act]F \wedge \langle Act\rangle \mathit{true}$$

Material for the Semantics

- This set of slides!
- Chapters 5 and 6 of book 'Reactive Systems – Modelling, Specification and Verification' by L. Aceto, A. Ingólfsdóttir, K. Larsen and J. Srba
- Section 6.4 of the book (and possibly Section 15.3)

Outline

Semantics of HML

With each formula associate a set of states where the formula is valid.

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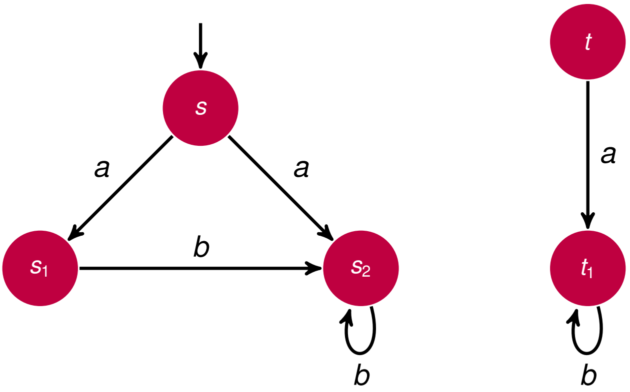
6. $[\![[a]F]\!] = [\cdot a \cdot][\![F]\!]$
   where $[\cdot a \cdot]$ is defined by
   $[\cdot a \cdot]T = \{p \in S \mid \forall p'.\ p \stackrel{a}{\rightarrow} p' \Rightarrow p' \in T\}$

Examples



- $\langle \cdot a \cdot \rangle \{s_1, t_1\} = \{s, t\}$

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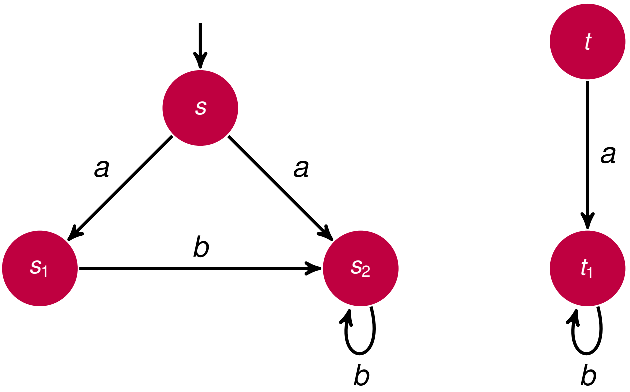
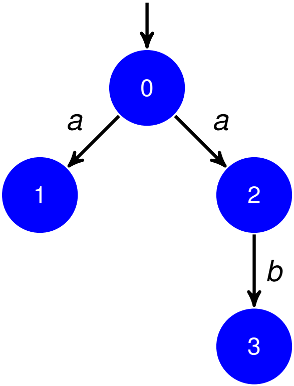


- $\langle \cdot a \cdot \rangle \{s_1, t_1\} = \{s, t\}$
- $[\cdot a \cdot]\{s_1, t_1\} = \{s_1, s_2, t, t_1\}$

Is the HML formula $\langle a\rangle\langle b\rangle true$ satisfied by the labeled transition system (i.e., by its initial state)?



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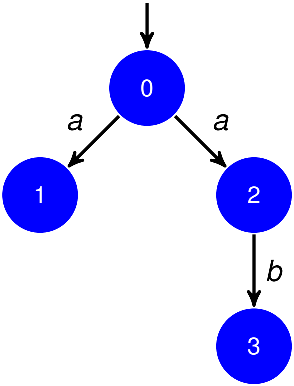

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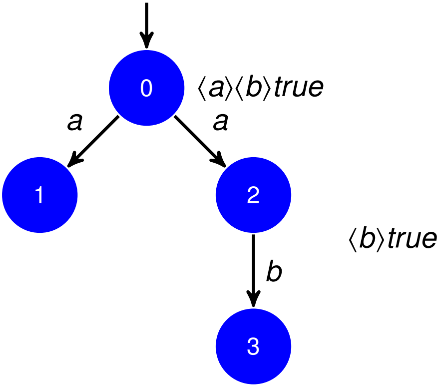

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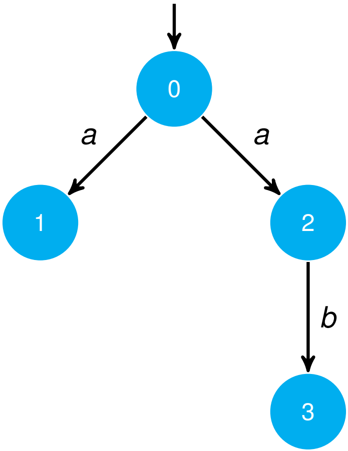

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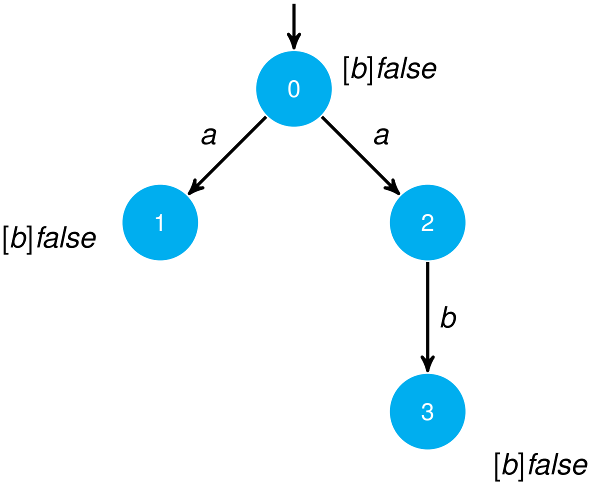

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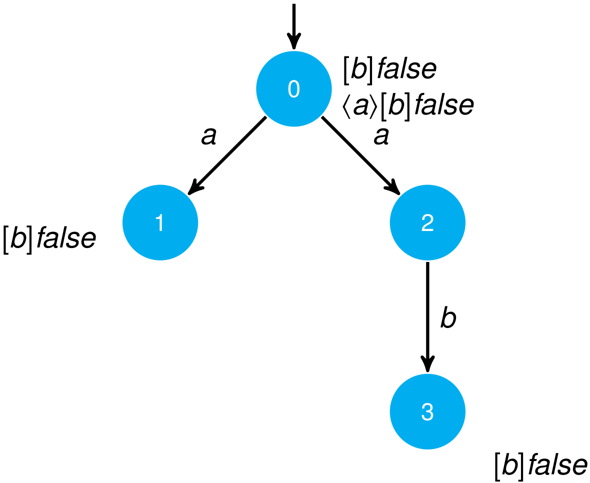

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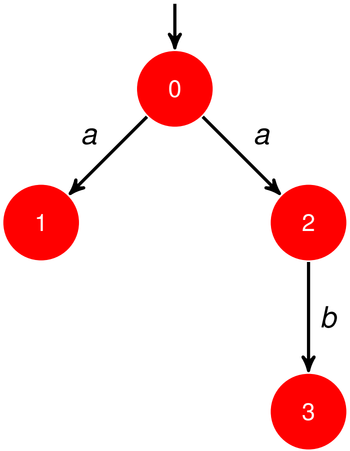

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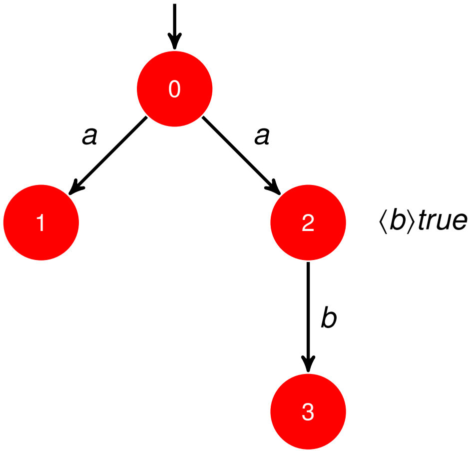

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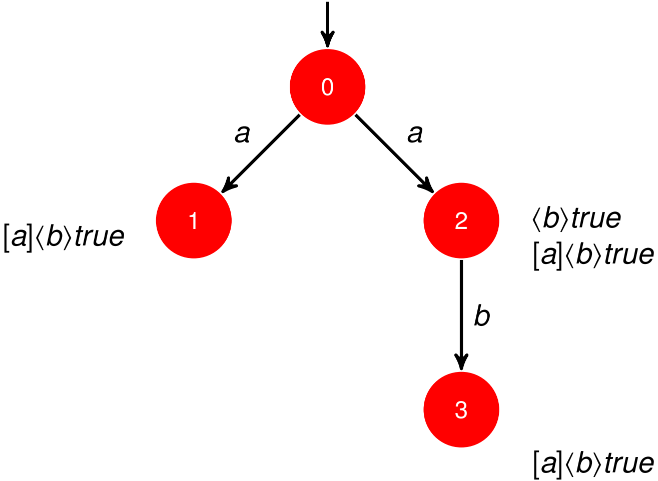

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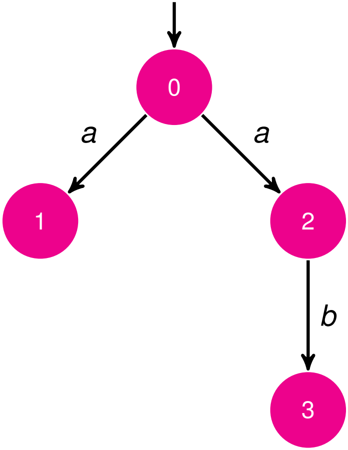

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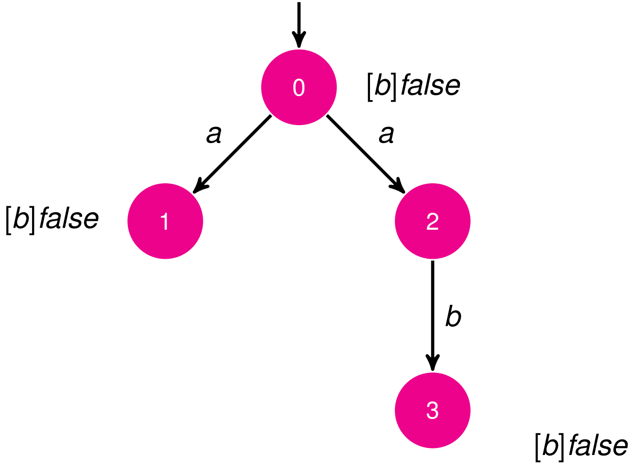

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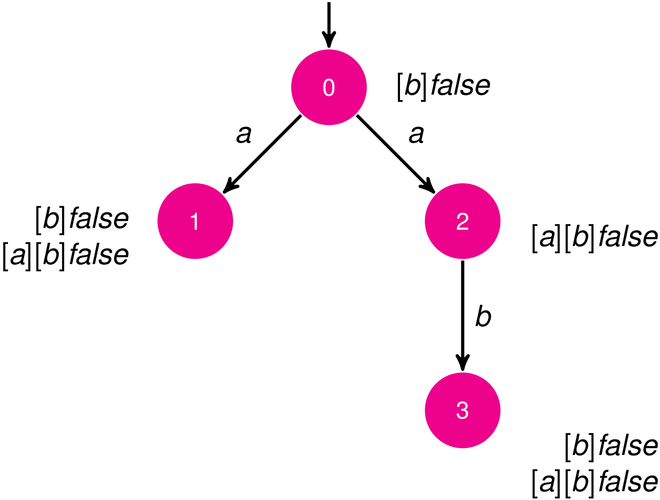

Outline

Temporal logic

Hennessy-Milner logic

Semantics of HML

Recursion

Semantics of Recursion

Limitations of HML

Limited expressiveness of HML

Using Hennessy-Milner Logic we can only describe properties of behaviors with a finite depth.

Modal depth

- $md(true) = md(false) = 0$
- $md(F \wedge G) = md(F \vee G) = \max\{md(F), md(G)\}$
- $md([a]F) = md(\langle a \rangle F) = md(F) + 1$

Temporal Properties not Expressible in HML

- $Inv(F)$ iff all reachable states satisfy F

$$Inv(F) = F \wedge [Act]F \wedge [Act][Act]F \wedge [Act][Act][Act]F \wedge \ldots$$

- $Pos(F)$ iff there is a reachable state which satisfies F

$$Pos(F) = F \vee \langle Act\rangle F \vee \langle Act\rangle\langle Act\rangle F \vee \langle Act\rangle\langle Act\rangle\langle Act\rangle F \vee \ldots$$

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Problems

- infinite formulae are not allowed in HML
- infinite formulae are difficult to handle

Why not to use recursion?

- $Inv(F)$ expressed by $X \stackrel{\text{def}}{=} F \wedge [Act]X$
- $Pos(F)$ expressed by $X \stackrel{\text{def}}{=} F \vee \langle Act \rangle X$

Why not to use recursion?

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Recursion on natural numbers

$$n \ : \ n \stackrel{\text{def}}{=} n^2$$

$$n \ : \ n \stackrel{\text{def}}{=} n + 1$$

$$n \ : \ n \stackrel{\text{def}}{=} 1 \times n$$

HML with one recursively defined variable

Syntax of Formulae

Formulae are given by the following abstract syntax

$$F ::= X \mid \mathit{true} \mid \mathit{false} \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid \langle a \rangle F \mid [a]F$$

where $a \in \mathit{Act}$ and X is a distinguished variable with a definition

- $X \stackrel{\text{min}}{=} F_X$, or $X \stackrel{\text{max}}{=} F_X$

such that F_X is a formula of the logic (which can contain X).

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Alternative syntax:

$$\begin{array}{lll} F & ::= & X \mid \mathit{true} \mid \mathit{false} \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid \langle a \rangle F \mid [a]F \\ & \mid & \mu X.F \mid \nu X.F \end{array}$$

Example:

$$X \stackrel{\text{min}}{=} X$$

Any set of states S satisfies the set-equation $X = X$. The least such set is $\emptyset$.

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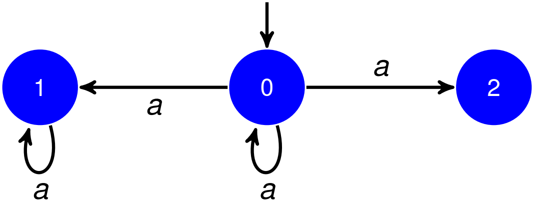
$$X \overset{\text{max}}{=} X$$

Any set of states S satisfies the set-equation $X = X$. The greatest such set is S.

Eventually 'a' will be disabled:

$$X \stackrel{?}{=} [a]\mathit{false} \vee \langle Act \rangle X$$



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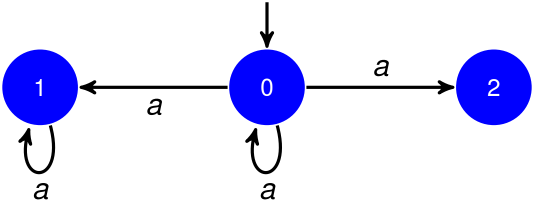

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Solutions of this equation are the sets: $\{0,2\}$ and $\{0,1,2\}$

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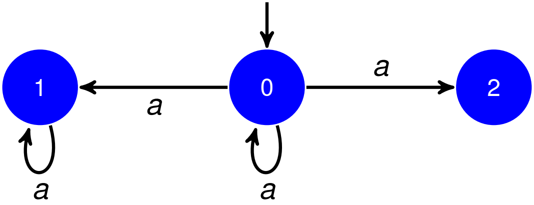

The property is valid for the labeled transition system
Solutions of this equation are the sets: $\{0, 2\}$ and $\{0, 1, 2\}$
We intended to describe the least solution!

$$X \stackrel{\min}{=} [a]\mathit{false} \vee \langle \mathit{Act} \rangle X$$

Example: A state can be reached where *a* cannot be executed:

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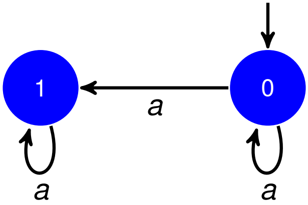

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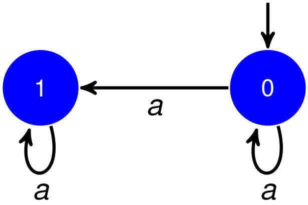
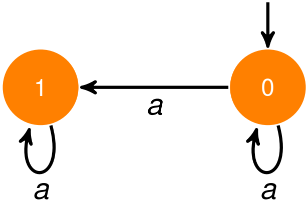


The unique least solution for this equation is the set of states $\emptyset$

Hence the property is not valid for the labeled transition system

Example: In every reachable state an *a*-transition is possible

$$X \stackrel{?}{=} \langle a \rangle \mathit{true} \wedge [\mathit{Act}]X$$



Solutions: $\emptyset$, $\{1\}$, and $\{0, 1\}$

Example: In every reachable state an *a*-transition is possible

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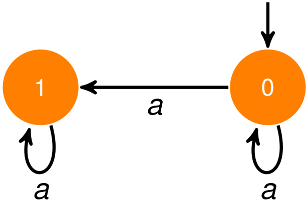

Solutions: $\emptyset$, $\{1\}$, and $\{0, 1\}$

We intended to describe the greatest solution!

$$X \stackrel{\max}{=} \langle a \rangle true \wedge [Act]X$$

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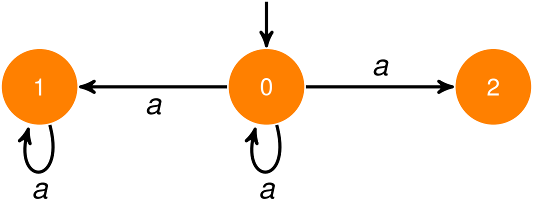

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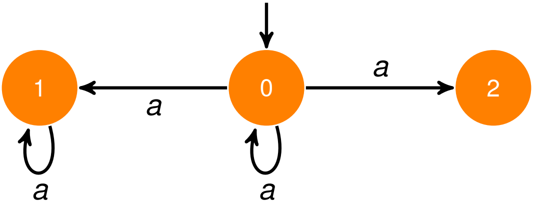
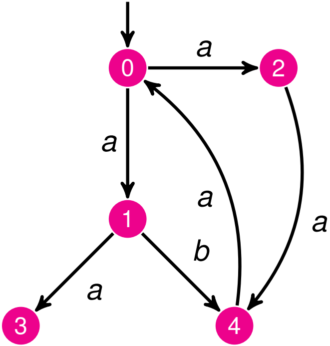


The greatest solution for this equation is the set of states $\{1\}$

Thus property is not valid for the labeled transition system

$$X \stackrel{\min}{=} \langle b \rangle \mathit{true} \vee \langle Act \rangle X$$

There is a path to a state where a *b* is possible



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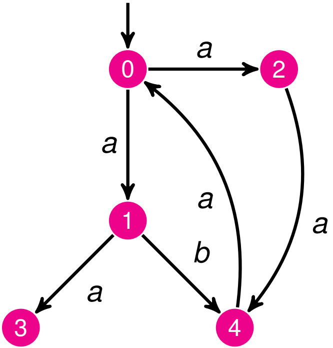

The least solution is the set of states $\{0, 1, 2, 4\}$; thus, property is valid for the labeled transition system

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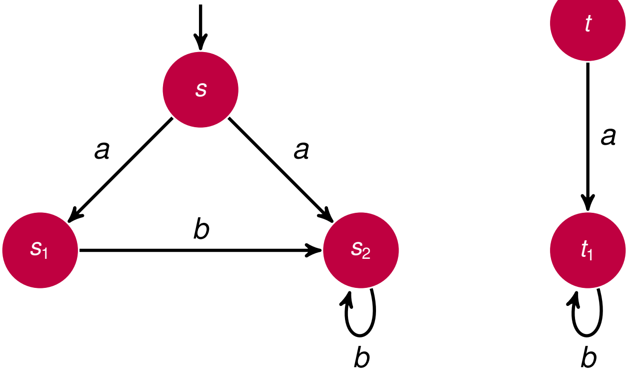
initially and after each b, one can take a b transition



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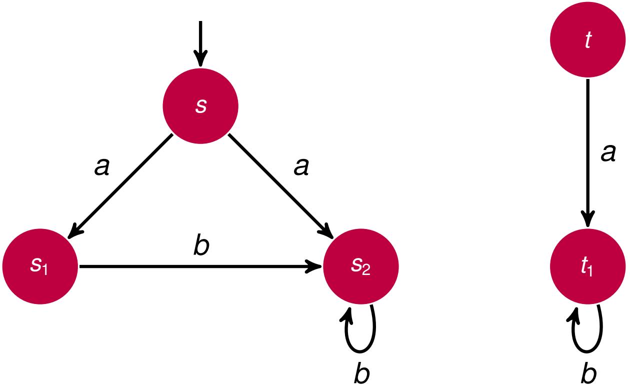

The greatest solution is the set of states $\{s_1, s_2, t_1\}$.

Formulas for the properties that cannot be expressed in HML

- the scientist never drinks beer

$$X \stackrel{\max}{=} [\text{beer}]\mathit{false} \wedge [\mathit{Act}]X$$

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Outline

Semantics of Recursion (one variable)

- With each formula associate a set of states for which it is satisfied

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$$[\![F]\!] \subseteq S$$
- How to deal with recursion variable X?
- Make an assumption on states satisfied by X. For every formula F we define a function $O_F : 2^S \to 2^S$ s.t.
  - if S is the set of processes that satisfy X
  - then $O_F(S)$ is the set of processes that satisfy F.

Definition of $O_F : 2^{\mathcal{S}} \to 2^{\mathcal{S}}$

For $S \subseteq \mathcal{S}$

$$
\begin{aligned}
O_X(S) &= S \\
O_{true}(S) &= \mathcal{S} \\
O_{false}(S) &= \emptyset \\
O_{F_1 \wedge F_2}(S) &= O_{F_1}(S) \cap O_{F_2}(S) \\
O_{F_1 \vee F_2}(S) &= O_{F_1}(S) \cup O_{F_2}(S) \\
O_{\langle a \rangle F}(S) &= \langle \cdot a \cdot \rangle O_F(S) \\
O_{[a]F}(S) &= [\cdot a \cdot] O_F(S)
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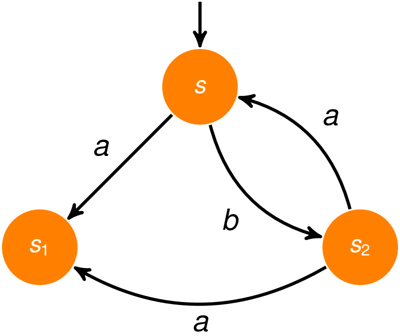

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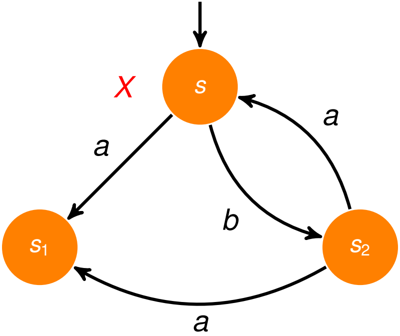

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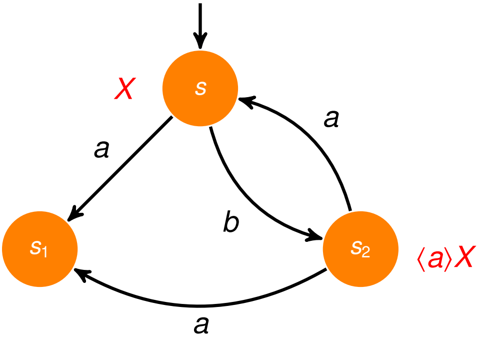

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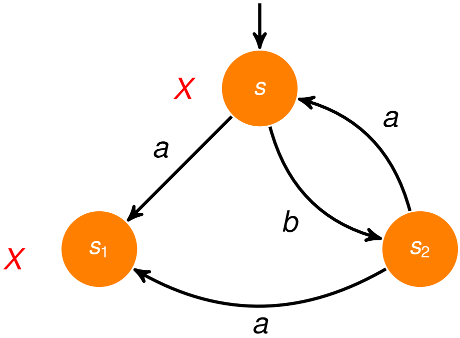

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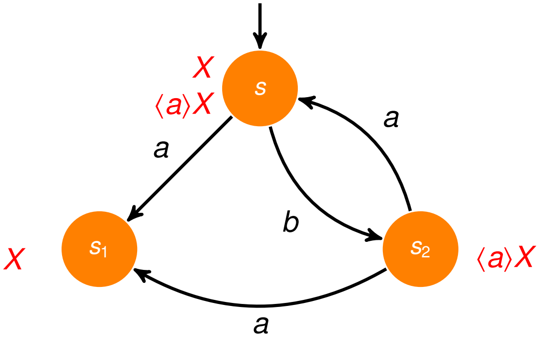

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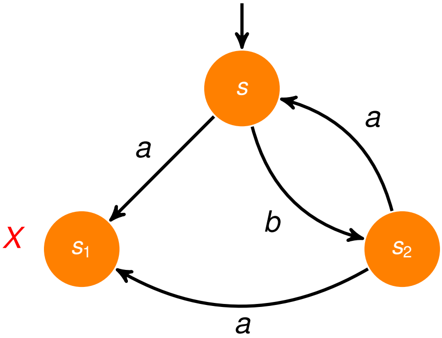

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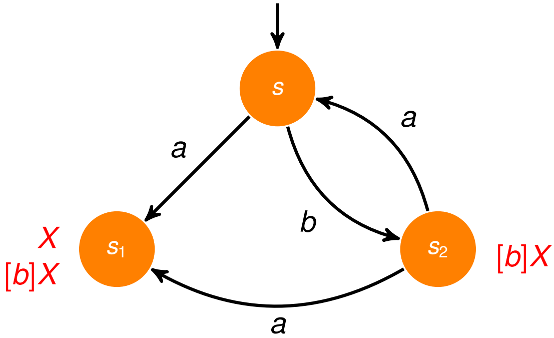

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5. $[\![\langle a \rangle F]\!] = \langle \cdot a \cdot \rangle [\![F]\!]$ where $\langle \cdot a \cdot \rangle : 2^S \to 2^S$ is defined by

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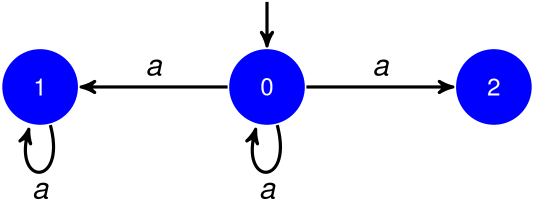
8. If $X \stackrel{\max}{=} F_X$ then $[\![X]\!] = \bigcup\{S \subseteq S \mid S = O_{F_X}(S)\}$

Let S be a finite set.

$$\frac{\text{Computing the solution of } X \stackrel{\min}{=} F_X}{\text{There exists a natural number } m > 0 \text{ such that } [\![X]\!] = {O_{F_X}}^m(\emptyset)}$$

$$\frac{\text{Computing the solution of } X \stackrel{\max}{=} F_X}{\text{There exist a natural number } M > 0 \text{ such that } [\![X]\!] = {O_{F_X}}^M(S)}$$

Example: $X \stackrel{\min}{=} [a]\mathit{false} \vee \langle \mathit{Act} \rangle X$



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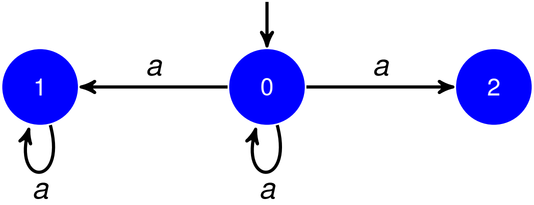

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\begin{aligned}
O_{F_X}(S) &= O_{[a]\mathit{false}}(S) \cup O_{\langle Act \rangle X}(S) \\
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\end{aligned}
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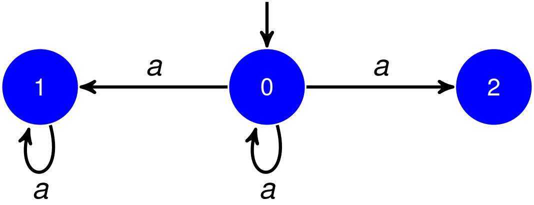

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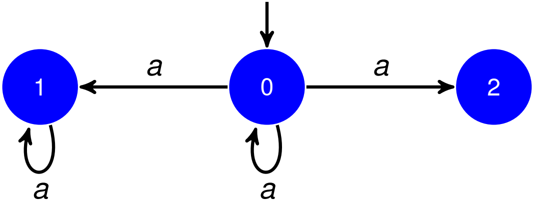

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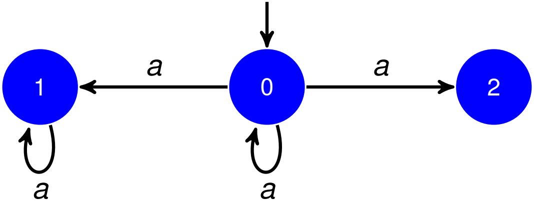

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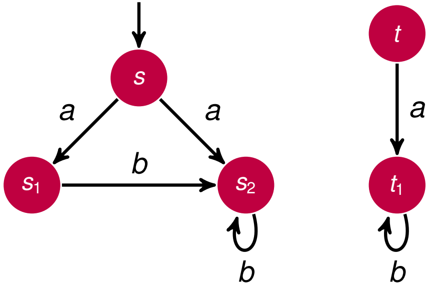

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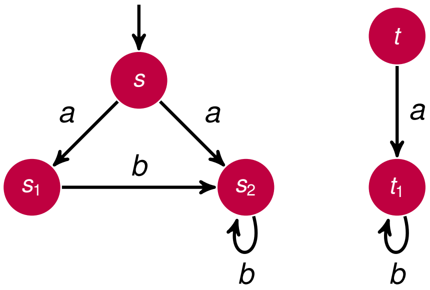

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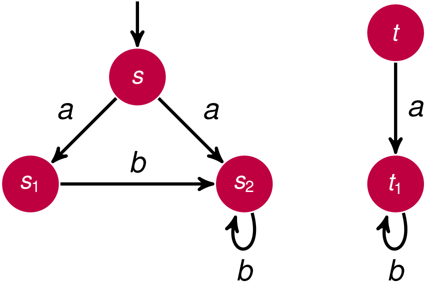

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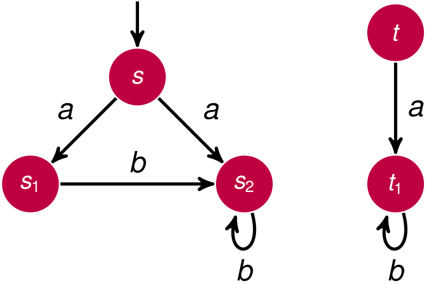

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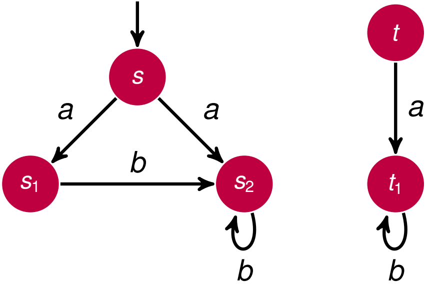

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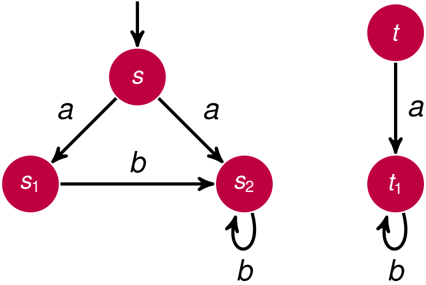

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Some temporal properties

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- $F \, \mathcal{U}^s \, G$: sooner or later G holds and until then F holds in all states traversed

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Some temporal properties

Using until we can express e.g. $\mathit{Inv}(F)$ and $\mathit{Even}(F)$:

$\mathit{Inv}(F)$ and $F\, \mathcal{U}^w\, \mathit{false}$ are logically equivalent

$\mathit{Even}(F)$ and $\mathit{true}\, \mathcal{U}^s\, F$ are logically equivalent