Bending Deflection – Statically Indeterminate Beams

AE1108-II: Aerospace Mechanics of Materials

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Recap

Procedure for Statically Indeterminate Problems

I. Free Body Diagram

II. Equilibrium of Forces (and Moments)

III. Displacement Compatibility

IV. Force-Displacement (Stress-Strain) Relations

Solve when number of equations = number of unknowns

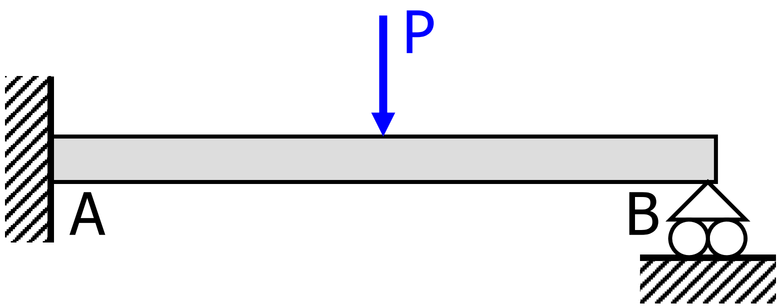
V. Answer the Question! – Typically calculate desired internal stresses, relevant displacements, or failure criteria

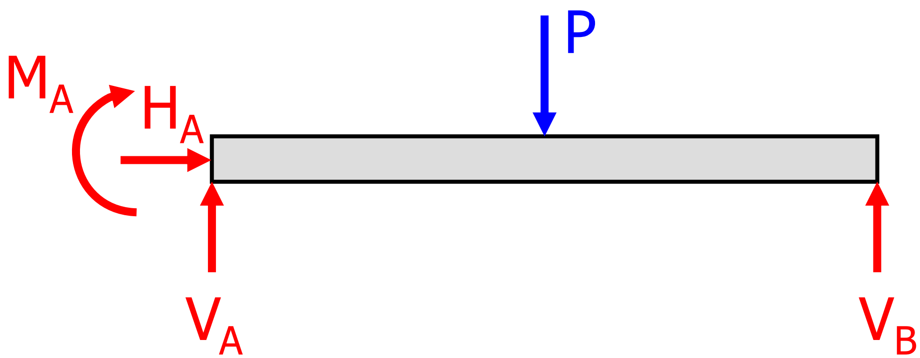
For bending, Force-Displacement relationships come from Moment-Curvature relationship
(ie: use Method of Integration or Method of Superposition)

Statically Indeterminate Beams

Many more redundancies are possible for beams:

- Draw FBD and count number of redundancies
- Each redundancy gives rise to the need for a compatibility equation





- 4 reactions
- 3 equilibrium equations

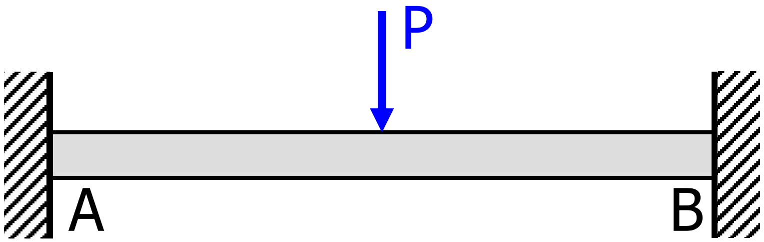
4 – 3 = 1

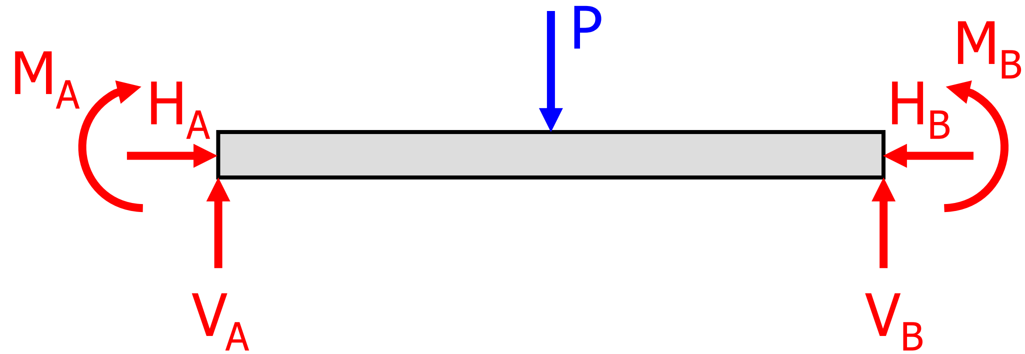
1st degree statically indeterminate

Statically Indeterminate Beams

Many more redundancies are possible for beams:

- Draw FBD and count number of redundancies
- Each redundancy gives rise to the need for a compatibility equation





- 6 reactions
- 3 equilibrium equations

$6 - 3 = 3$

3rd degree statically indeterminate

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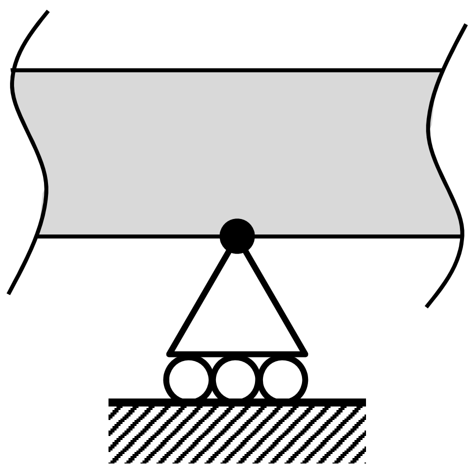
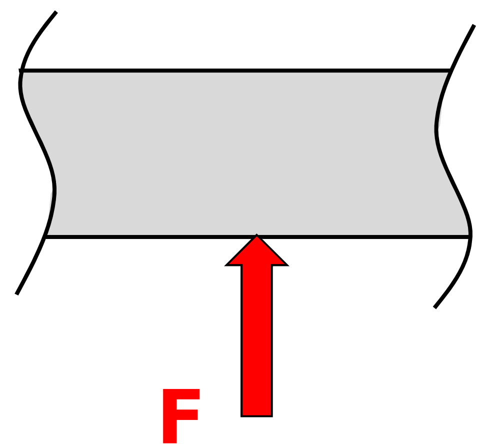
Solving statically indeterminate beams using method of integration

What is the difference between a support and a force?

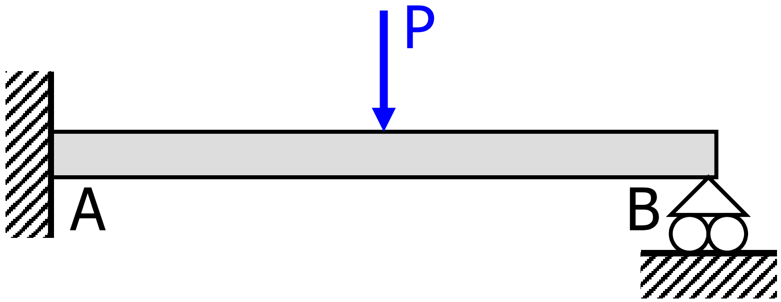




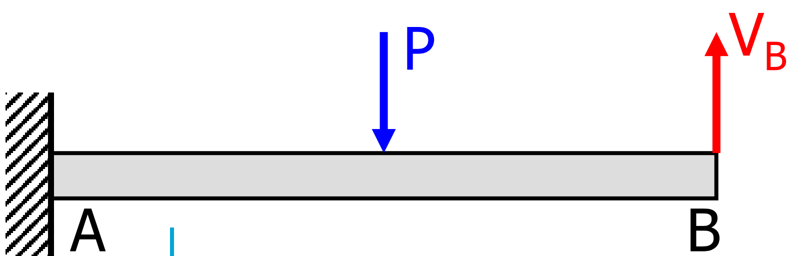
Displacement Compatibility

(support places constraint on deformation)

Method of Integration



What if we remove all redundancies and replace with reaction forces?



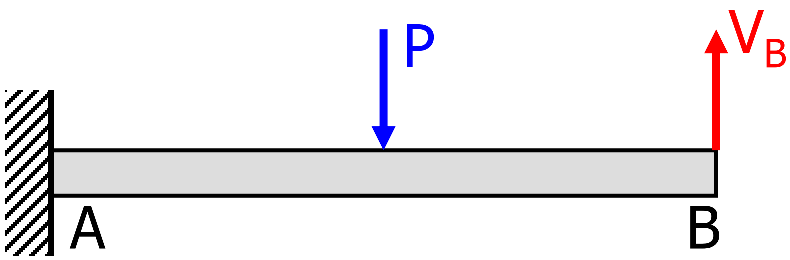
If we treat V_B as known, we can solve!

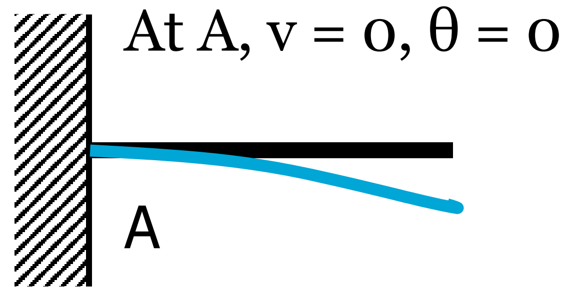
Formulate expression for M(z)

$$M = -EI\frac{d^2v}{dz^2}$$

Can integrate to find v

Method of Integration (cont)





$$EI\frac{d^2v}{dz^2} = -M\left(z, V_B\right)$$

$\int$

$$\left.\begin{array}{l} EIv' = \int M\left(z,V_B\right)\cdot dz \xrightarrow{\qquad} C_1 \\ \\ EIv = \iint M\left(z,V_B\right)\cdot dz \xrightarrow{\qquad} C_2 \end{array}\right\} \text{Determine constants of integration from Boundary Conditions}$$

$\int$

$\boxed{v(z, V_B)}$ How to determine V_B?

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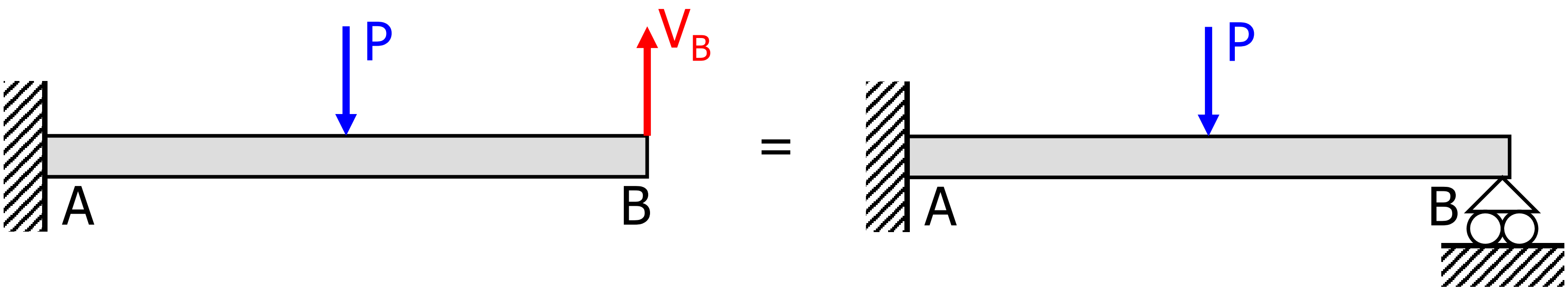
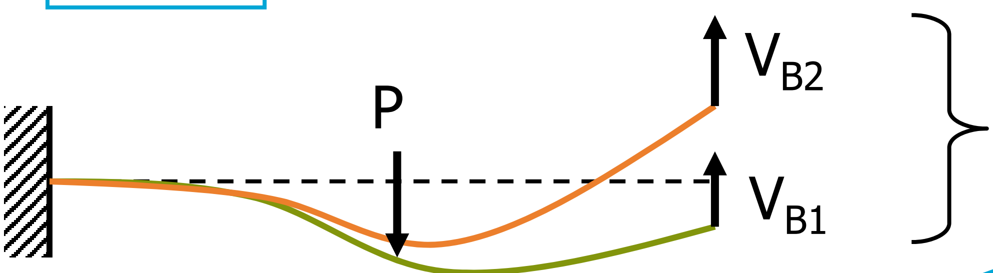
Method of Integration (cont)



$\boxed{v(z, V_B)}$

How to determine V_B?



V_B changes displacement!

Must be compatible!

Compatibility BC: At B, v = 0

Solve for V_B

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Compatibility equations for beams are simply the boundary conditions at redundant supports

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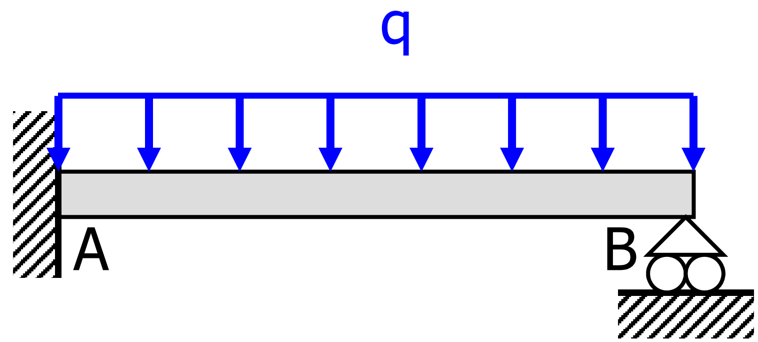
Example 1

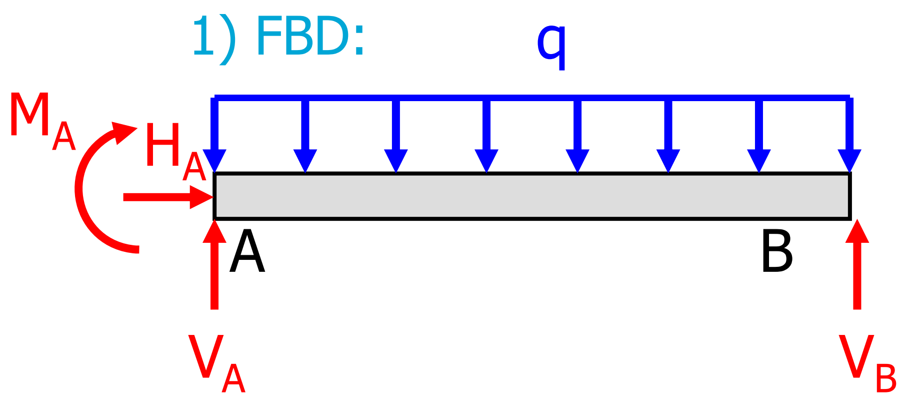
Problem Statement

Determine deflection equation for the beam using method of integration:



Solution

1) FBD:



2) Equilibrium:

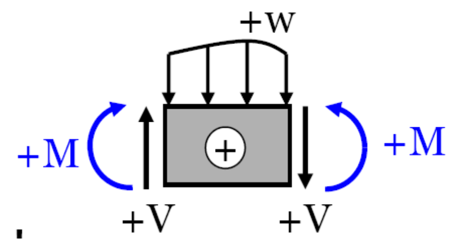
$$\sum \overset{\rightarrow +}{F} \Rightarrow H_A = 0$$

$$\sum F^{\overset{+}{\uparrow}} \Rightarrow V_A + V_B = qL$$

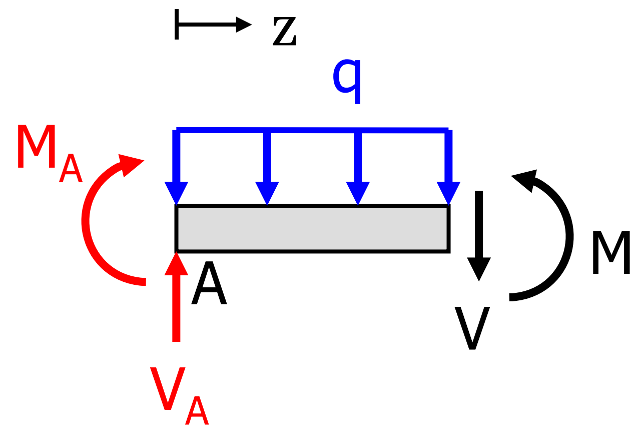
$$\sum M_A \Rightarrow M_A = LV_B - \frac{qL^2}{2} \quad = \frac{qL^2}{2} - LV_A$$

Treat reaction forces as knowns!

Example 1



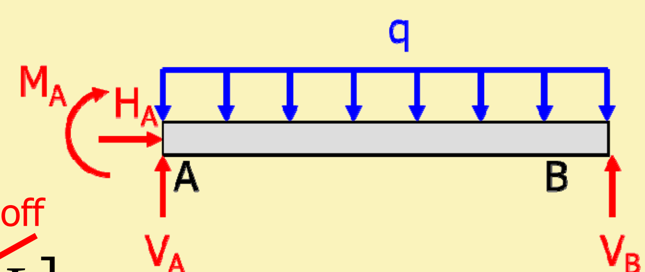
4) Determine moment equation:



$$\sum M_z^{ccw+} \Rightarrow M = M_A + V_A z - (qz)\frac{z}{2}$$

$$= M_A + V_A z - \frac{q}{2} z^2$$

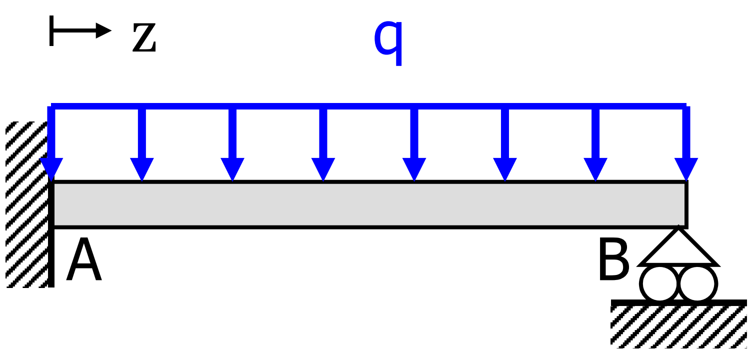
Can also use step function approach



$$M = \overbrace{M_A[z-0]^0 + V_A[z-0] - \frac{q}{2}[z-0]^2}^{\text{always on}} + \overset{\text{always off}}{\cancel{V_B[z-L]}}$$

Example 1

5) Integrate Moment equation to get v′ and v



$$\int \quad -EIv'' = M_A + V_A z - \frac{q}{2} z^2 \quad = M(z)$$

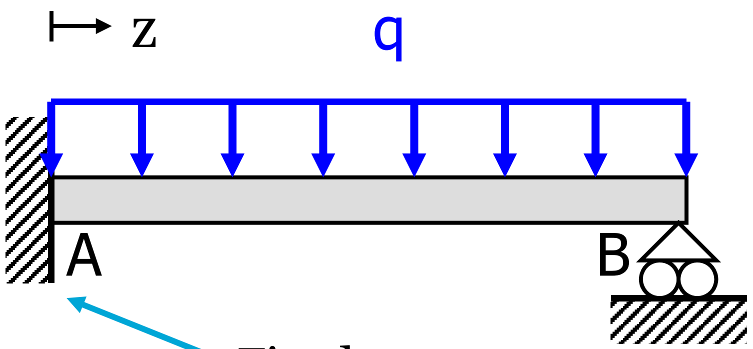
$$\int \quad -EIv' = M_A z + \frac{V_A}{2} z^2 - \frac{q}{6} z^3 + C_1 \quad = -EI\theta(z)$$

$$-EIv = \frac{M_A}{2} z^2 + \frac{V_A}{6} z^3 - \frac{q}{24} z^4 + C_1 z + C_2 \quad = -EIv(z)$$

We now have expressions for v and v′, but need to determine constants of integration and unknown reactions

Example 1

5a) Solve for Constants of Integration using BC's:



$$-EIv' = M_A z + \frac{V_A}{2}z^2 - \frac{q}{6}z^3 + \cancel{C_1}^{\,0} = -EI\theta(z)$$

$$-EIv = \frac{M_A}{2}z^2 + \frac{V_A}{6}z^3 - \frac{q}{24}z^4 + \cancel{C_1}^{\,0} z + C_2 = -EIv(z)$$

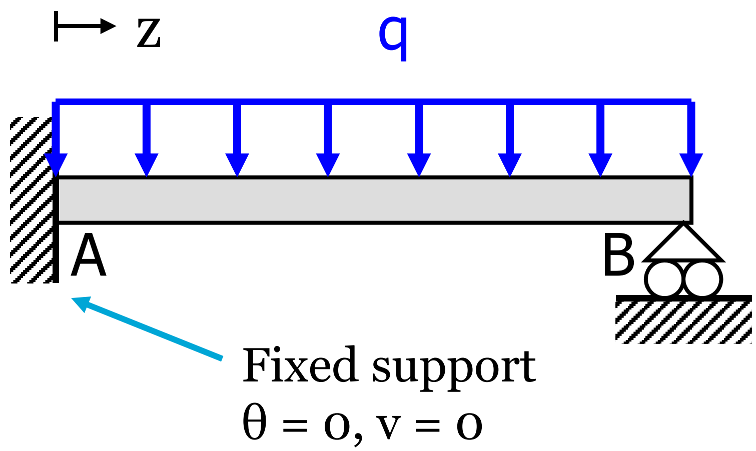
<u>Boundary Conditions:</u>

At z = 0, θ = 0 $\quad \Rightarrow -EI(0) = -\frac{q}{6}(0)^3 + \frac{V_A}{2}(0)^2 + M_A(0) + C_1$

$$\therefore C_1 = 0$$

Example 1

5a) Solve for Constants of Integration using BC's:



$$-EIv' = M_A z + \frac{V_A}{2} z^2 - \frac{q}{6} z^3 + \cancelto{0}{C_1} = -EI\theta(z)$$

$$-EIv = \frac{M_A}{2} z^2 + \frac{V_A}{6} z^3 - \frac{q}{24} z^4 + \cancelto{0}{C_1} z + \cancelto{0}{C_2} = -EIv(z)$$

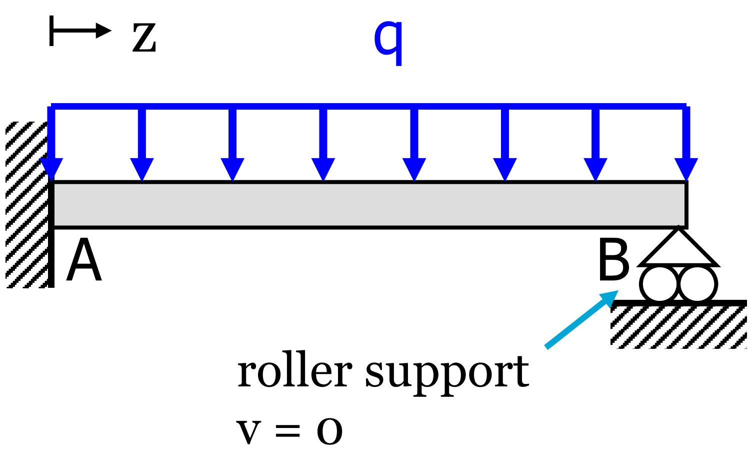
<u>Boundary Conditions:</u>

At z = 0, v = 0 $\quad \Rightarrow -EI(0) = -\frac{q}{24}(0)^4 + \frac{V_A}{6}(0)^3 + \frac{M_A}{2}(0)^2 + C_2$

$$\therefore C_2 = 0$$

Example 1

5b) Solve for Reaction Forces using BC's (imposed by redundant support):



$$-EIv' = M_A z + \frac{V_A}{2} z^2 - \frac{q}{6} z^3$$ = -EIθ(z)

$$-EIv = \frac{M_A}{2} z^2 + \frac{V_A}{6} z^3 - \frac{q}{24} z^4$$ = -EIv(z)

Boundary Conditions:

At z = L, v = 0

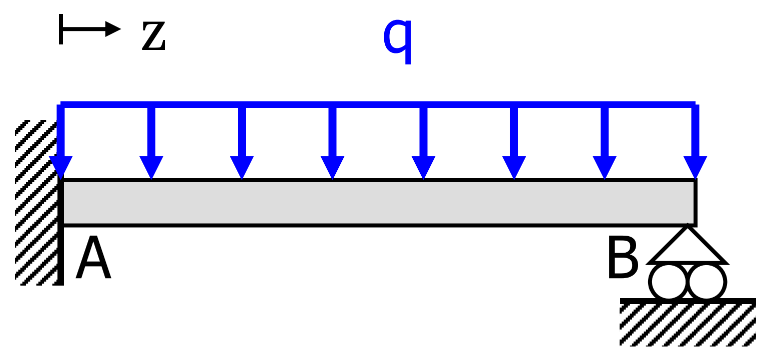
$$\Rightarrow -EI(0) = -\frac{q}{24}(L)^4 + \frac{V_A}{6}(L)^3 + \frac{M_A}{2}(L)^2$$

$$V_A = \frac{qL}{4} - \frac{3M_A}{L}$$

Recall from equilibrium:

$$M_A = \frac{qL^2}{2} - LV_A \qquad V_B = qL - V_A$$

Example 1



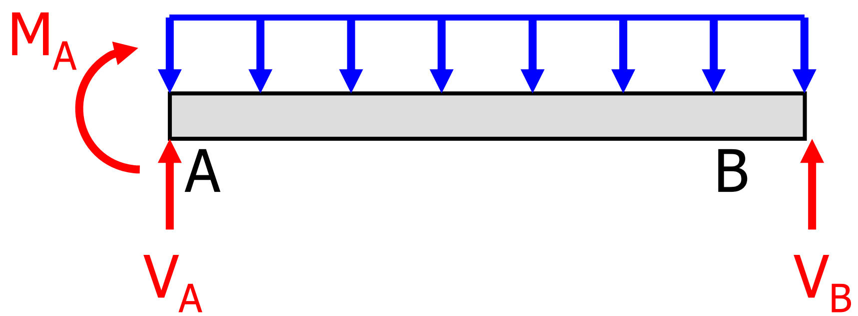
5b) Solve for Reaction Forces using BC's (imposed by redundant support):

$$\underbrace{V_A = \frac{qL}{4} - \frac{3M_A}{L},\quad M_A = \frac{qL^2}{2} - LV_A,\quad V_B = qL - V_A}$$

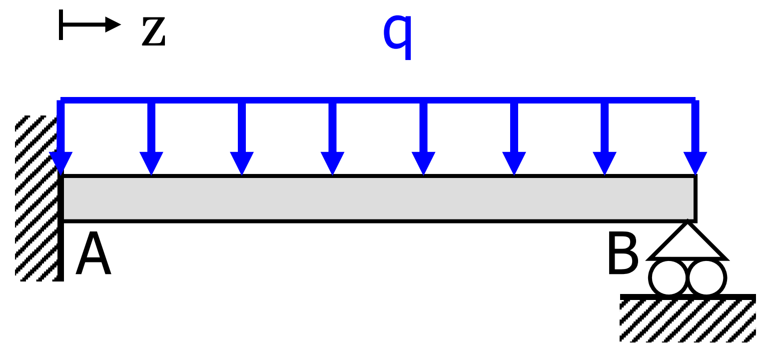
$$V_A = \frac{5}{8}qL$$

$$M_A = -\frac{qL^2}{8}$$

$$V_B = \frac{3}{8}qL$$



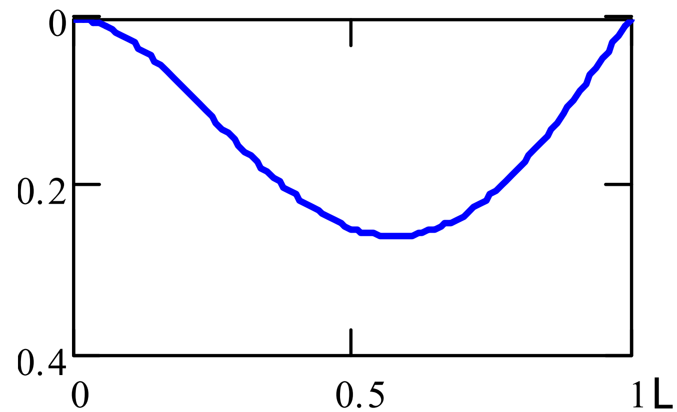
Example 1



We were asked to determine deflection equation:

$$-EIv = \frac{M_A}{2}z^2 + \frac{V_A}{6}z^3 - \frac{q}{24}z^4$$

$$v = \frac{qz^2}{48EI}\left(3L^2 - 5Lz + 2z^2\right)$$

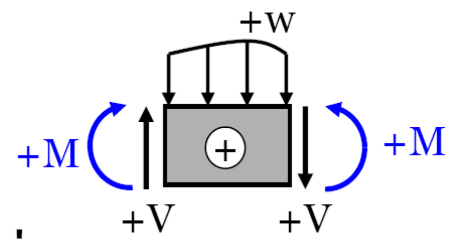


Max Displacement:

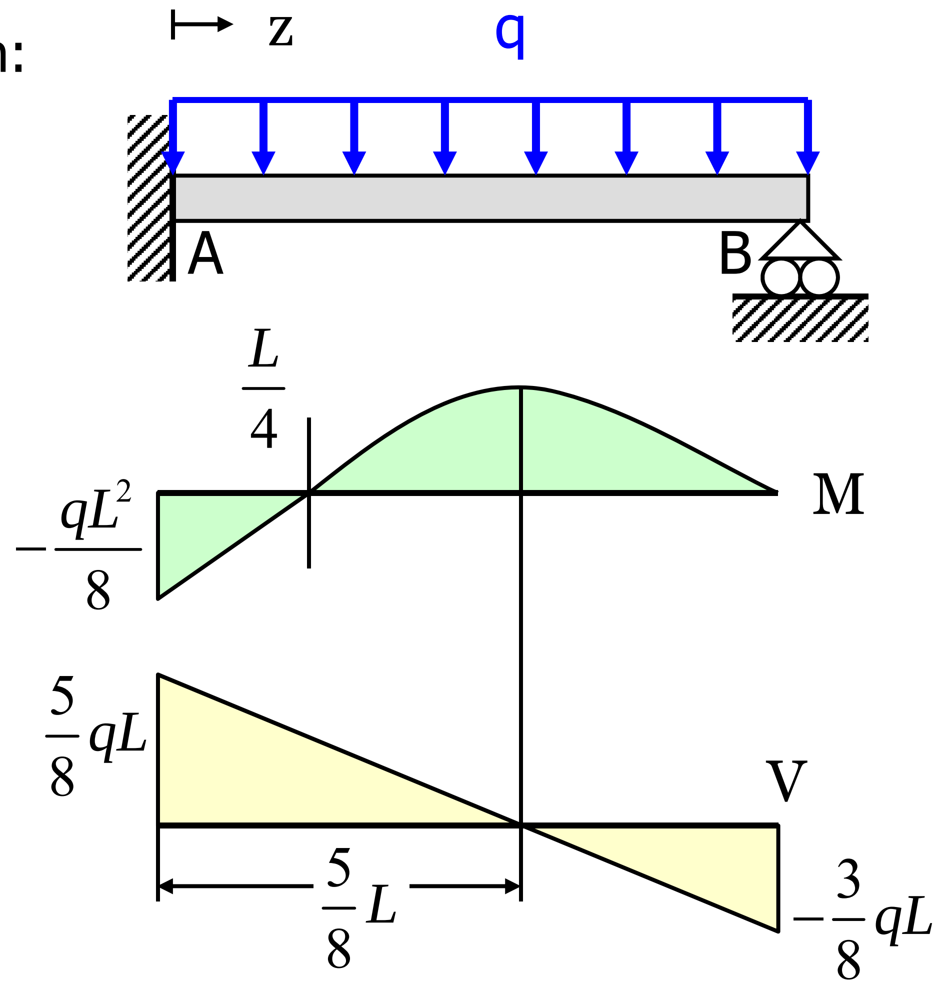
$$v' = \frac{q}{48EI}\left(\overbrace{6L^2z - 15Lz^2 + 8z^3}^{=0}\right) = 0 \quad \Rightarrow z \approx 0.5785L$$

$$v(0.5785L) = 0.005416\frac{qL^2}{EI}$$

Example 1



Now that the reactions are known:



$$M(z) = -EIv'' = M_A + V_A z - \frac{q}{2}z^2$$

$$= -\frac{qL^2}{8} + \frac{5qLz}{8} - \frac{qz^2}{2}$$

$$\frac{d}{dz}$$

$$V(z) = -EIv''' = \frac{5qL}{8} - qz$$

Solving statically indeterminate beams using superposition

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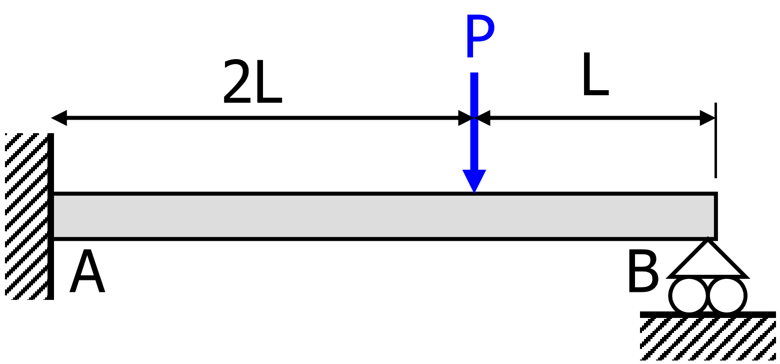
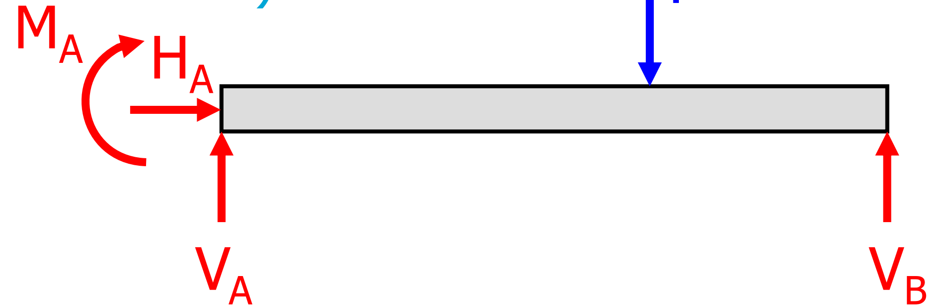
Method of Superposition

Determine reaction forces:



1) FBD:



- 4 reactions
- 3 equilibrium equations

1st degree statically indeterminate

2) Equilibrium:

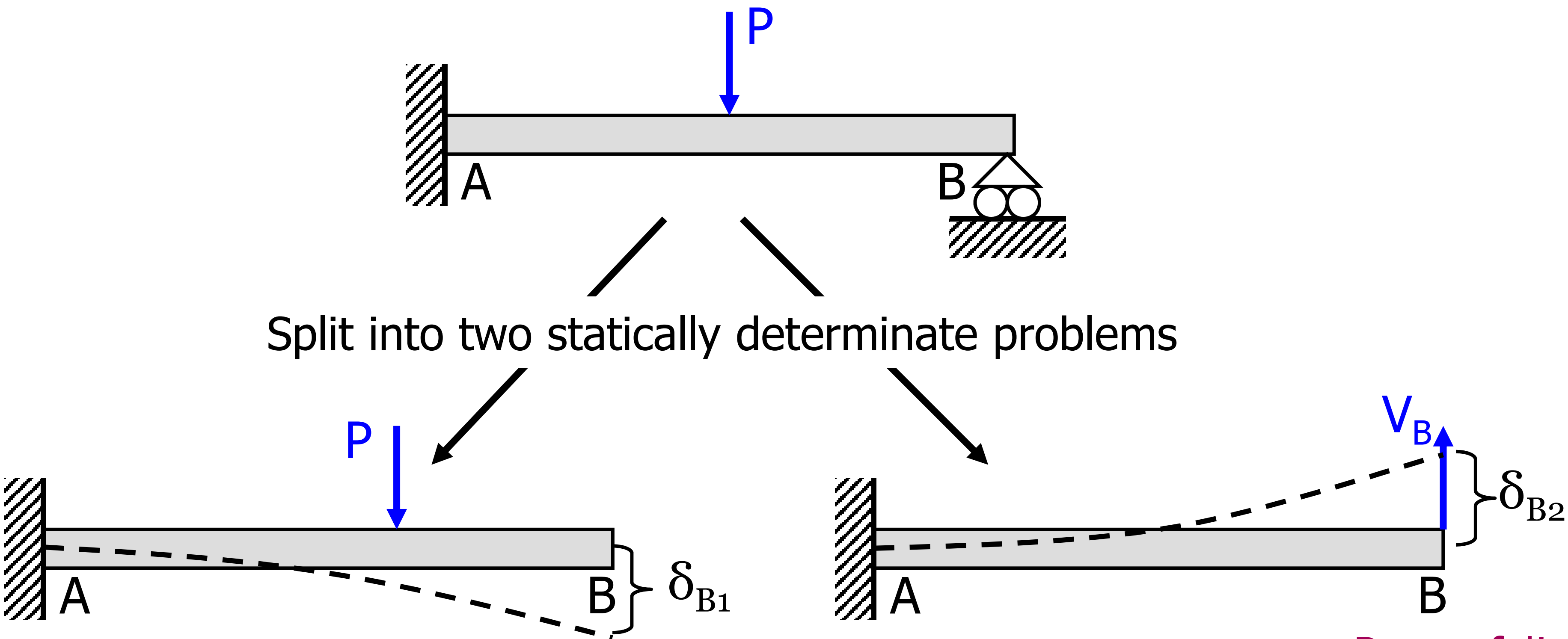
$$\sum \overset{\rightarrow +}{F} \Rightarrow H_A = 0$$

$$\sum F^{\overset{+}{\uparrow}} \Rightarrow V_A + V_B = P$$

$$\sum M_A \Rightarrow 3LV_B = 2LP + M_A$$

Method of Superposition (cont)

How do we get compatibility equation?



Be careful!

3) Compatibility: $\delta_{B1} = \delta_{B2}$ $\quad (v_{B1} + v_{B2} = 0)$ $\quad +\text{ve } v \downarrow$

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Method of Superposition (cont)

How do we get Force-Displacement relations?

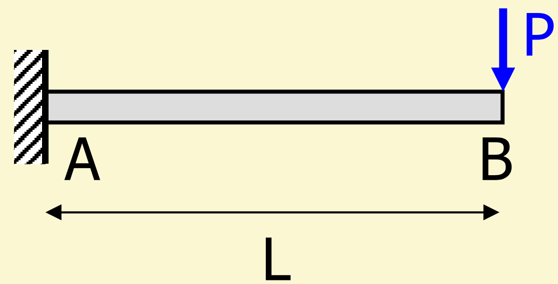
We have been doing this in the previous lectures

<u>Integrate Moment Curvature Relation</u>

$$\boxed{M = -EI\frac{d^2v}{dz^2}}$$

Can integrate to find v

<u>Standard Case Solutions</u>



$$\theta_B = \frac{PL^2}{2EI} \qquad v_B = \frac{PL^3}{3EI}$$

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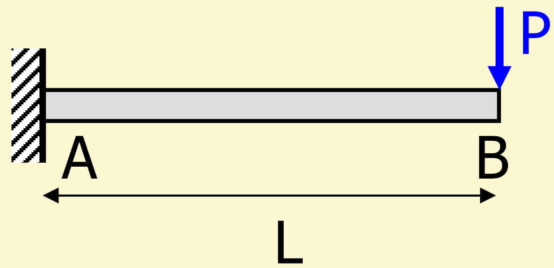
Method of Superposition (cont)

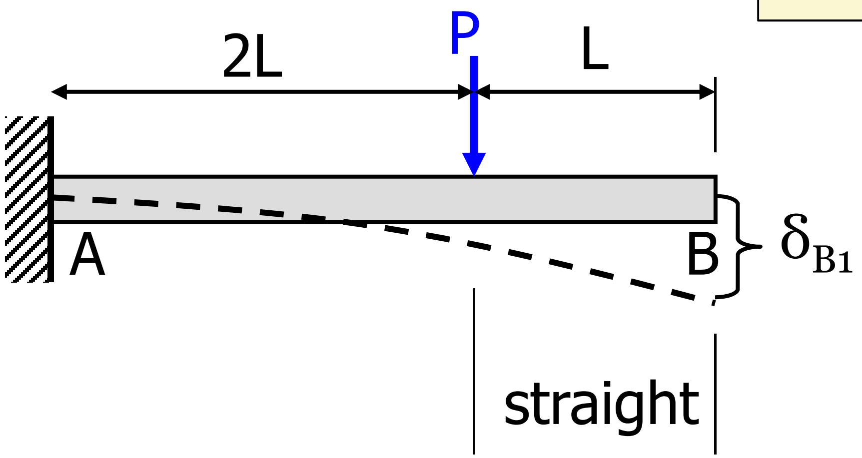
From the standard case:



$$\theta_B = \frac{PL^2}{2EI}$$

$$v_B = \frac{PL^3}{3EI}$$

4) Force-Displacement:



$$\theta_P = \frac{P\left(2L\right)^2}{2EI} = \frac{2PL^2}{EI}$$

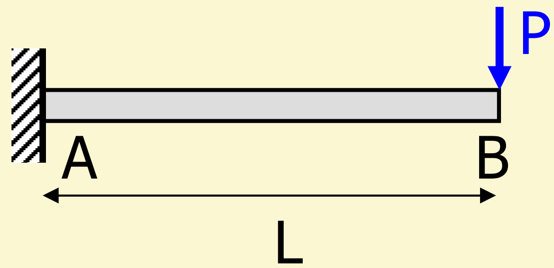
$$v_{B1} = v_P + \theta_P \cdot L$$

$$= \frac{P\left(2L\right)^3}{3EI} + \frac{2PL^2}{EI} \cdot L$$

$$= \frac{14PL^3}{3EI}$$

Method of Superposition (cont)

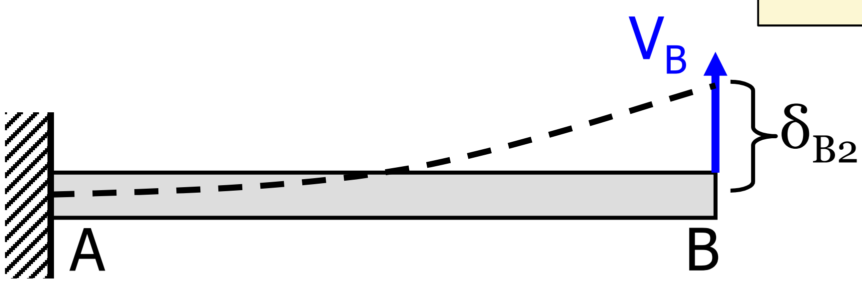
From the standard case:



$$\theta_B = \frac{PL^2}{2EI}$$

$$v_B = \frac{PL^3}{3EI}$$

4) Force-Displacement:



$$v_{B2} = \frac{-V_B (3L)^3}{3EI} = -\frac{9V_B L^3}{EI}$$

Compatibility:

$$v_{B1} + v_{B2} = 0 \quad = \frac{14PL^3}{3EI} + \left(-\frac{9V_B L^3}{EI} \right) \quad \Rightarrow V_B = \frac{14}{27} P$$

Additional remarks about bending deflections

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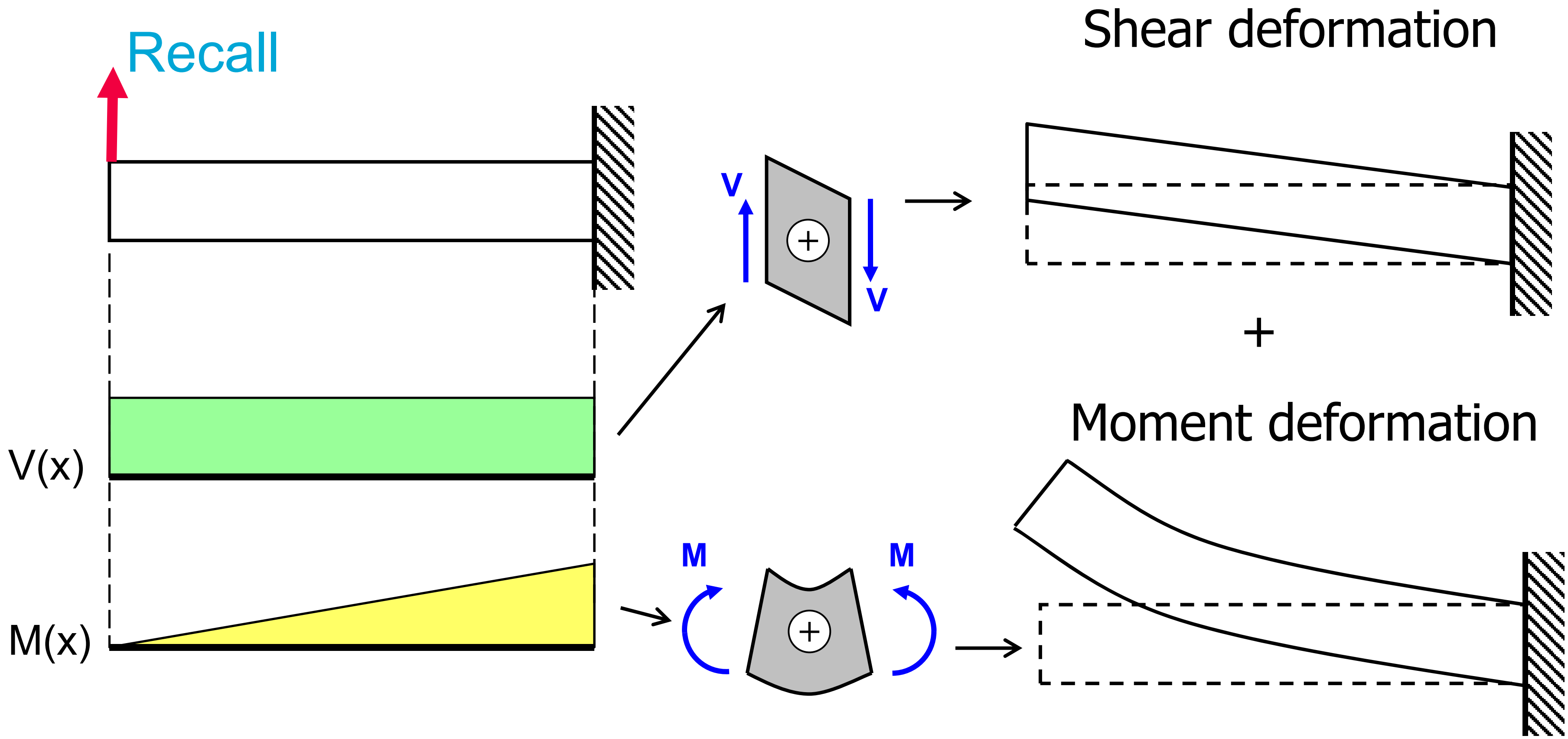
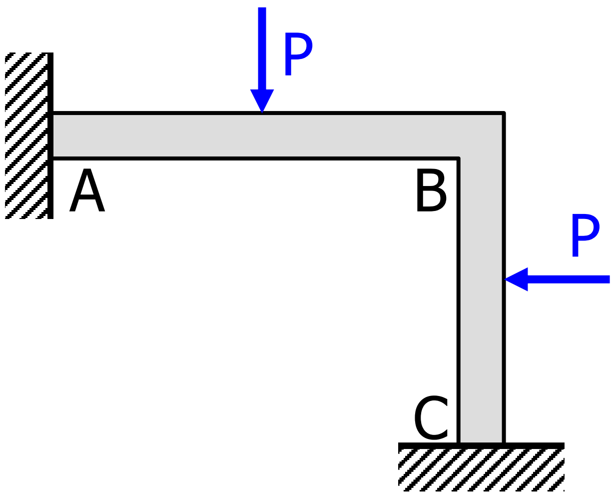
Remarks about Beam Deflections

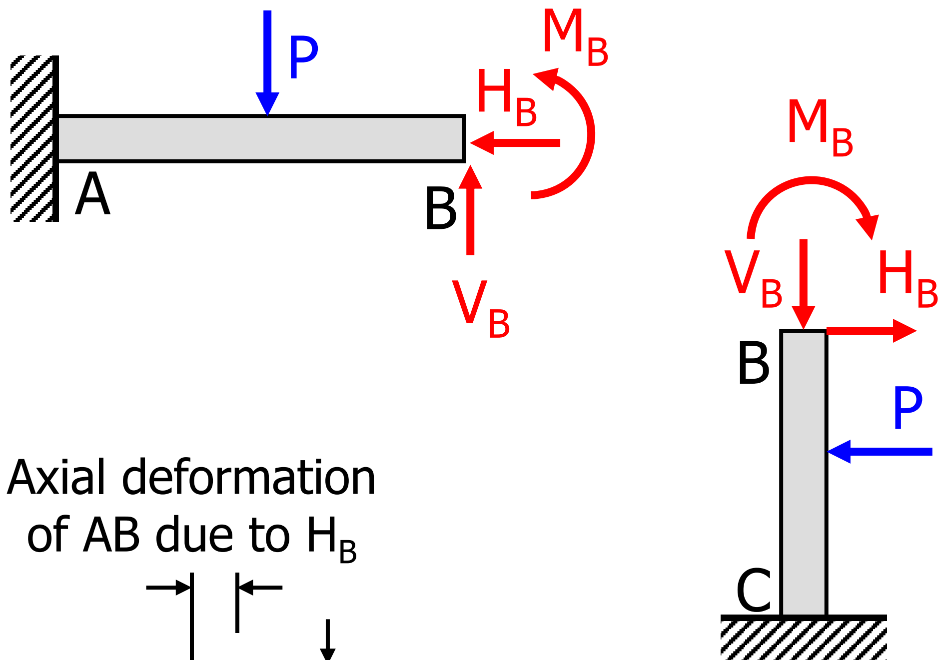


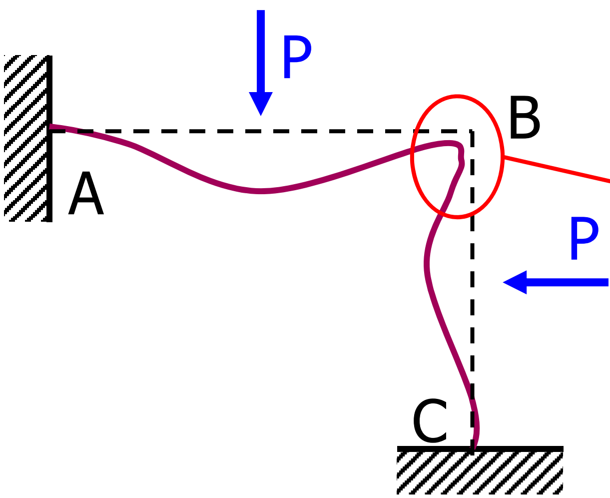
Negligible (for long beams)

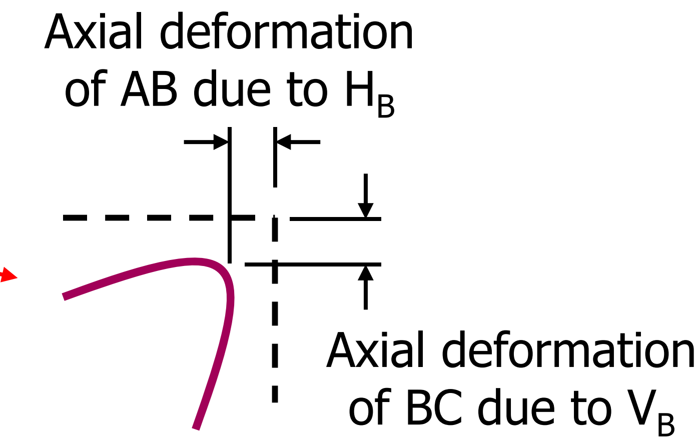
Bending Deformation = ~~Shear Deformation~~ + Moment Deformation

Remarks about Beam Deflections









Axial deformation << bending deformation!

Remark about Beam Deflections

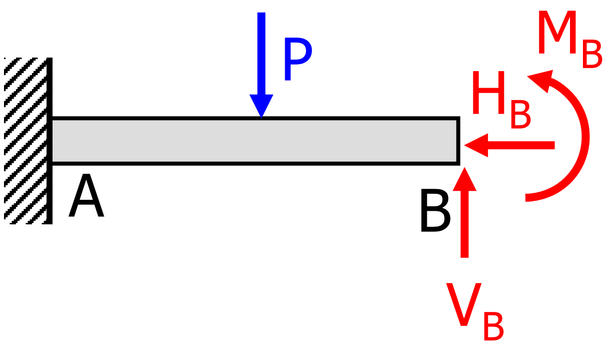
For bending deformation problems

negligible

Deformation = ~~Axial Deformation~~ + ~~Shear Deformation~~ + Moment Deformation

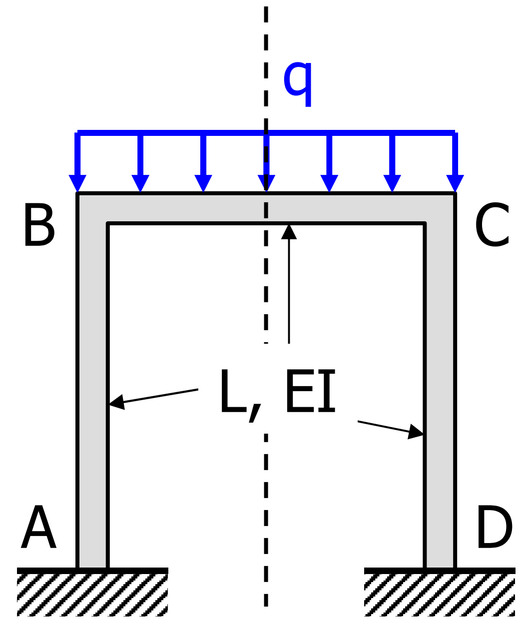
BUT!

If moment deformation is not present, deformation is not negligible



Example 2

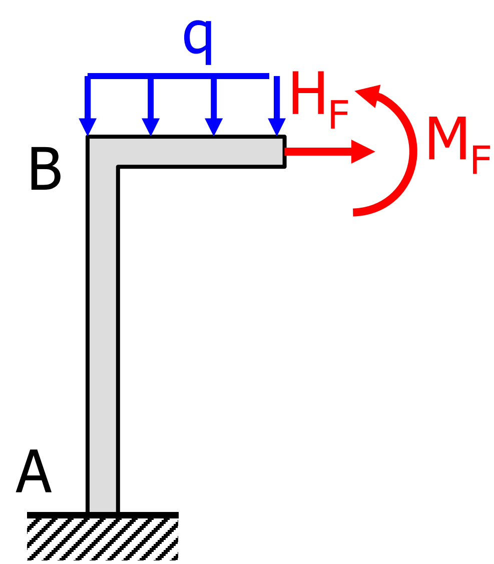
Calculate reaction forces at A and D:



First, can we see any simplifications?

Symmetry!

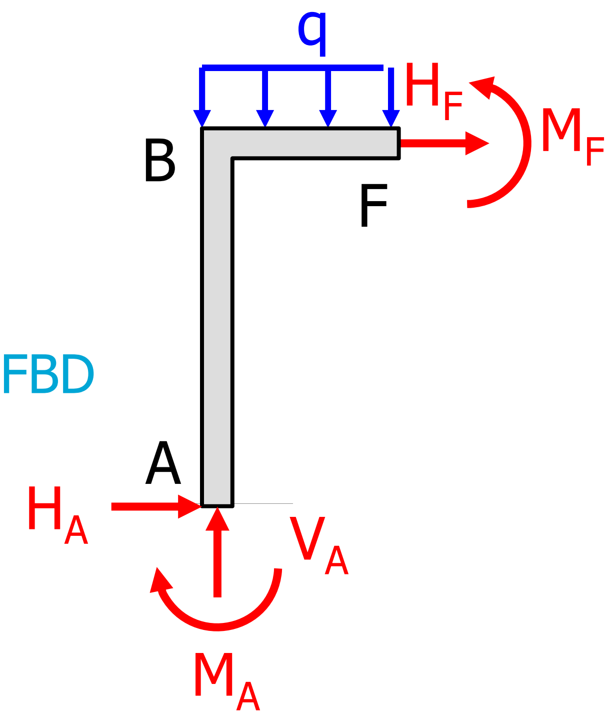
Symmetry implies:
- Reactions at A = reactions at D
- Slope at symmetry plane = 0
- Shear force at symmetry plane = 0



We will solve using superposition and standard cases

Example 2

Calculate reaction forces:



2) Equilibrium

$$\sum \overset{\rightarrow +}{F} \Rightarrow H_A = -H_F$$

$$\sum F^{\uparrow +} \Rightarrow V_A = \frac{qL}{2}$$

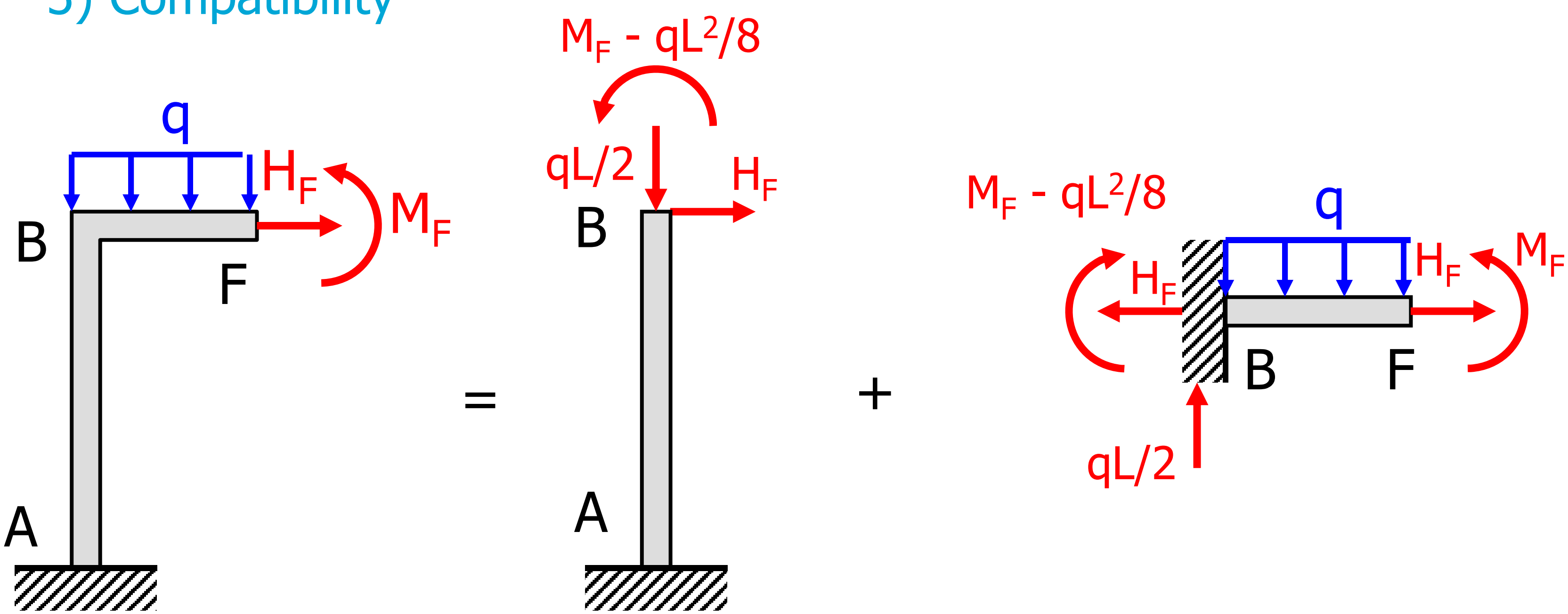
$$\sum M_A^{ccw+} \Rightarrow M_A + \frac{qL^2}{8} + H_F L = M_F$$

Split problem into two: beam AB and beam BF

Example 2

Calculate reaction forces:

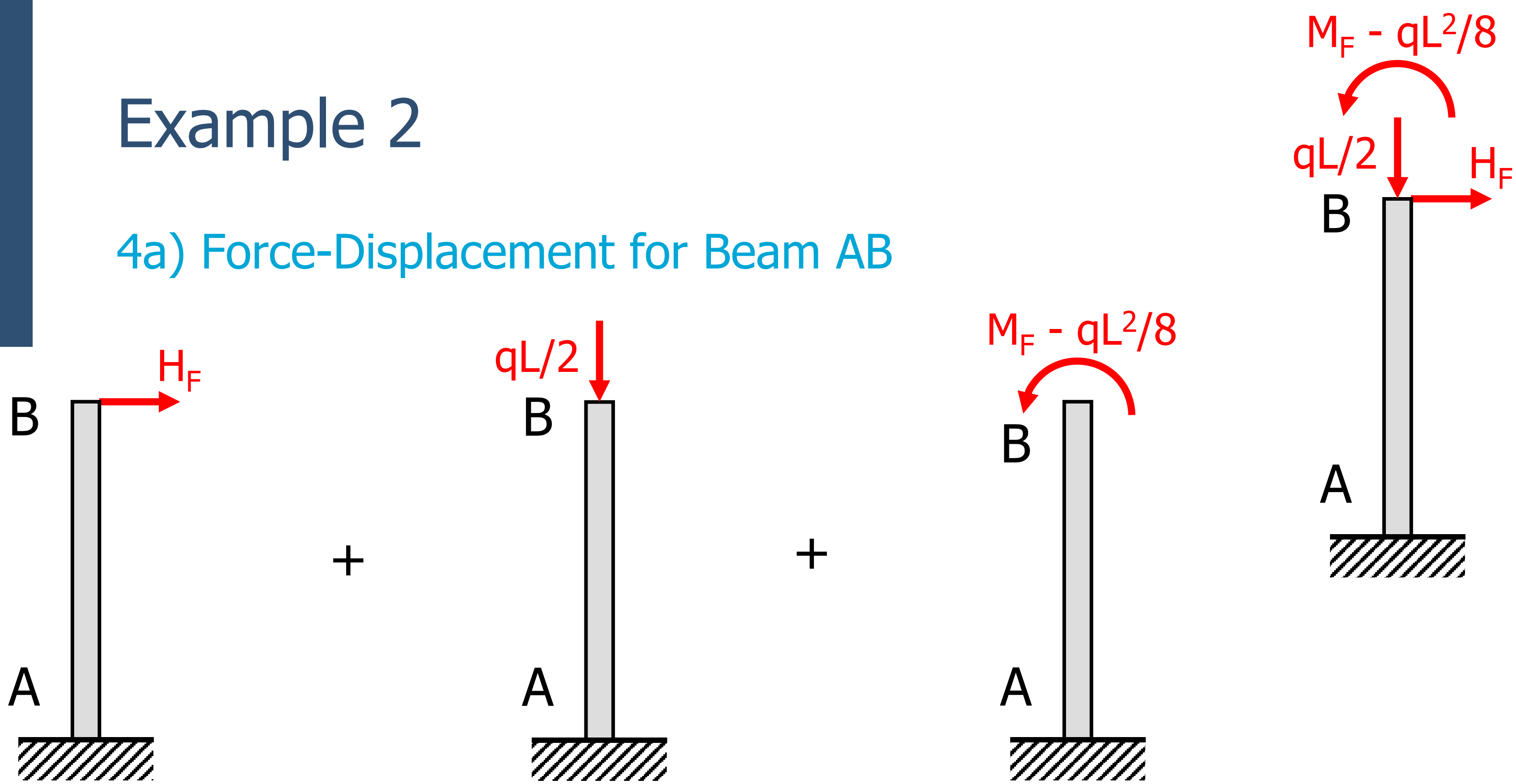
3) Compatibility



$$\theta_F = 0, \quad \underbrace{v_B = 0}$$ (axial deformation negligible)

Example 2

4a) Force-Displacement for Beam AB

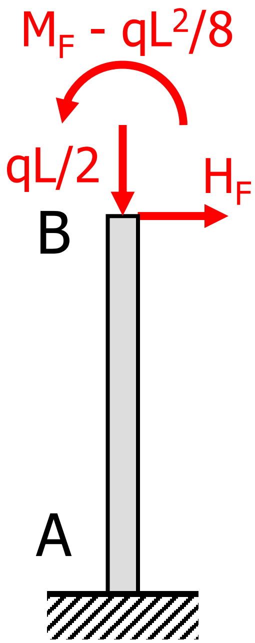


$$\overset{\rightarrow +}{v_B} = \frac{H_F L^3}{3EI} \qquad v_B = 0 \qquad \overset{\rightarrow +}{v_B} = -\frac{M_F L^2}{2EI} + \frac{qL^4}{16EI}$$

$$\theta_B^{cw+} = \frac{H_F L^2}{2EI} \qquad \theta_B = 0 \qquad \theta_B^{cw+} = -\frac{M_F L}{EI} + \frac{qL^3}{8EI}$$

Example 2

4a) Force-Displacement for Beam AB



$$\overset{\rightarrow +}{v_B} = \frac{H_F L^3}{3EI} - \frac{M_F L^2}{2EI} + \frac{qL^4}{16EI} \quad = 0$$

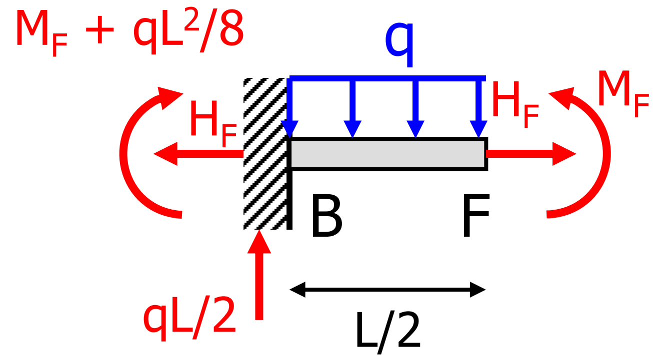
$$\theta_B^{cw+} = \frac{H_F L^2}{2EI} - \frac{M_F L}{EI} + \frac{qL^3}{8EI}$$

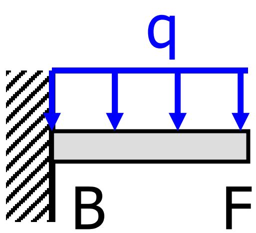
Recall compatibility:

Still need θ_F → $\theta_F = 0, \quad v_B = 0$

Example 2

4b) Force-Displacement for Beam BF





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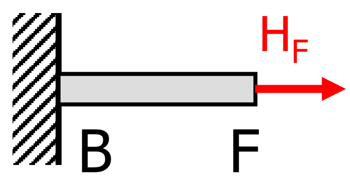

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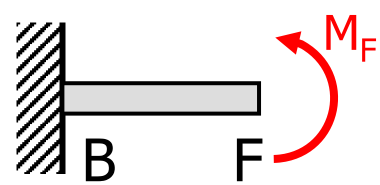


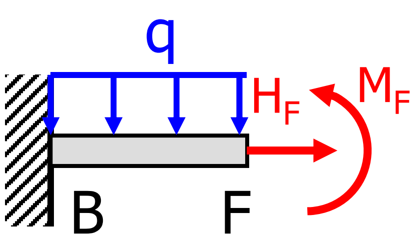
$$\theta_F^{cw+} = \frac{q\left(L/2\right)^3}{6EI} \qquad \theta_F = 0 \qquad \theta_F^{cw+} = -\frac{M_F L}{EI}$$

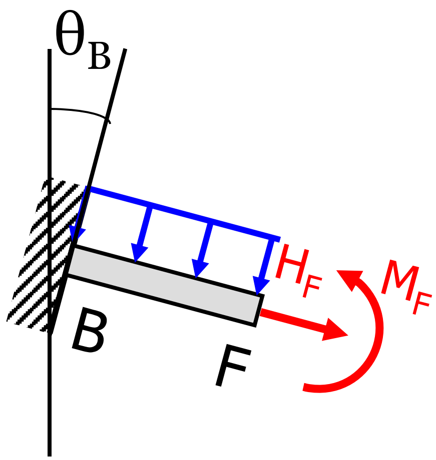
$$\theta_F^{cw+} = \frac{qL^3}{48EI} - \frac{M_F L}{EI}$$

Wait! Not entirely correct!

Example 2

Previous angle relative to fixed support B





$$\theta_F^{cw+} = \frac{qL^3}{48EI} - \frac{M_F L}{EI} + \theta_B$$

$$\theta_F^{cw+} = \frac{7qL^3}{48EI} - \frac{2M_F L}{EI} + \frac{H_F L^2}{2EI} \quad = 0$$

Example 2

Solve:

Recall compatibility:

$$\theta_F = 0, \quad \mathrm{v}_B = 0$$

$$\theta_F^{cw+} = \frac{7qL^3}{48EI} - \frac{2M_F L}{EI} + \frac{H_F L^2}{2EI} = 0$$

$$\overset{\rightarrow+}{v_B} = \frac{H_F L^3}{3EI} - \frac{M_F L^2}{2EI} + \frac{qL^4}{16EI} = 0$$

Recall equilibrium:

$$\sum \overset{\rightarrow+}{F} \Rightarrow H_A = -H_F$$

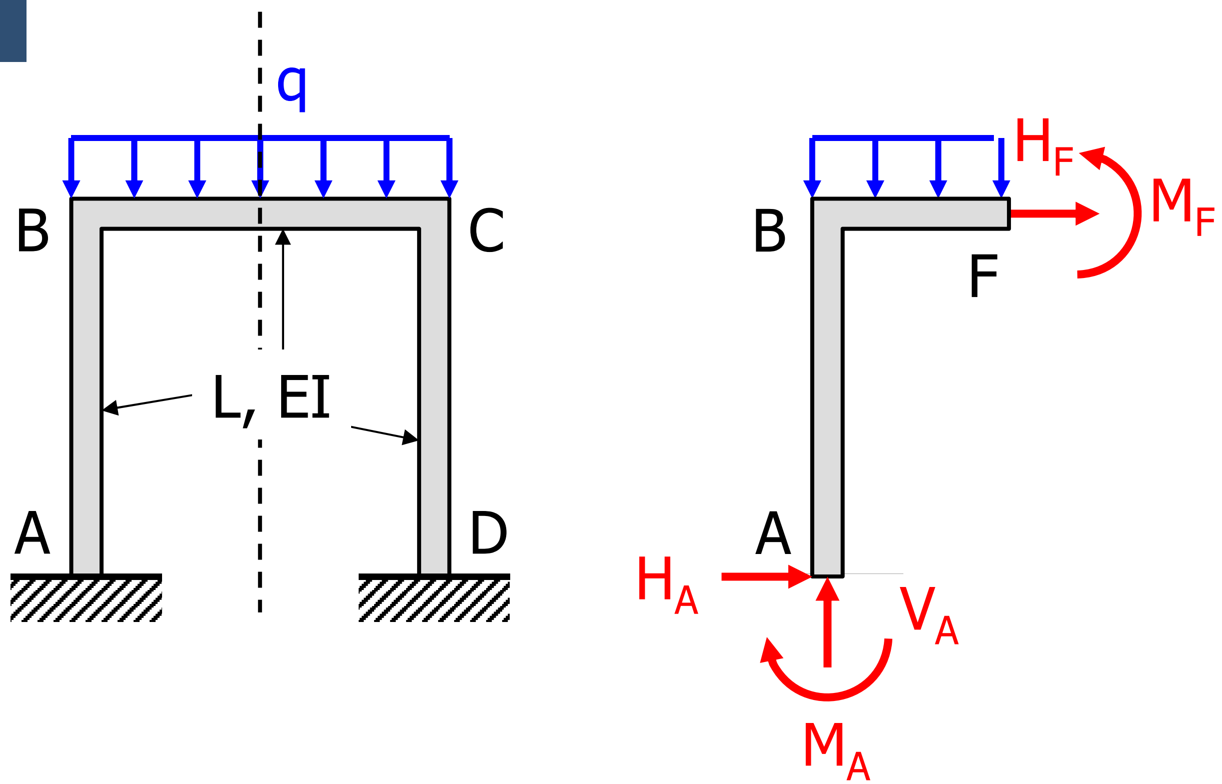
$$\sum F^{\uparrow+} \Rightarrow V_A = \frac{qL}{2}$$

$$\underbrace{\sum M_A^{ccw+} \Rightarrow M_A + \frac{qL^2}{8} + H_F L = M_F}$$

$$H_F = -\frac{qL}{8} \qquad H_A = \frac{qL}{8}$$

$$M_F = \frac{qL^2}{24} \qquad M_A = -\frac{2qL^2}{3}$$

Example 2



$$H_F = -\frac{qL}{8}$$

$$M_F = \frac{qL^2}{24}$$

$$H_A = \frac{qL}{8}$$

$$V_A = \frac{qL}{2}$$

$$M_A = -\frac{2qL^2}{3}$$

For next time

	First Lecture		Second Lecture		COZ		Instruction
Week	Topics	Preparation	Topics	Preparation	Assignment	Due Date	Activity
3.1	Introduction	None	Stress, Strain, Hooke's Law	L.U. 1 (all)	none	none	no Instruction
3.2	Axial loading and static indetermenacy	L.U. 2 (all)	Torsion of circular shafts	L.U. 3.1-3.2	COZ1	18/02/2016	Mock exam 1
3.3	Torsion of thin-walled shafts	L.U. 3.3	Bending stresses in beams	L.U. 4 (all)	COZ2	25/02/2016	Peer grading 1
3.4	Transverse shear stresses in beams	L.U. 5.1	Shear stresses in thin-walled beams	L.U. 5.2	COZ3	03/03/2016	Mock exam 2
3.5	Combined loading	L.U. 6 (all)	Stress transformations & Failure criteria	L.U. 7 (all)	COZ4	10/03/2016	Peer Grading 2
3.6	Beam deflections by integration	L.U. 8.1	Discontinuity functions and	L.U. 8.2-8.3	COZ5	17/03/2016	Mock exam 3
3.7	Statically indeterminate beams	L.U. 8.4	Review	None	COZ6	24/03/2016	Peer grading 3
3.8	Study for Exam						
3.9	Exam - Friday April 8th @ 13:30						

L.U. = Learning unit (refer to blackboard site for learning units)

TU Delft