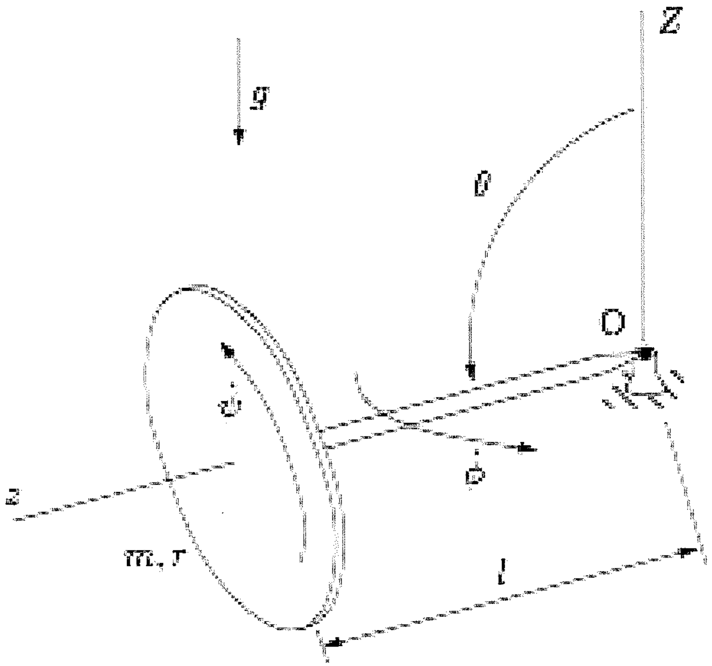
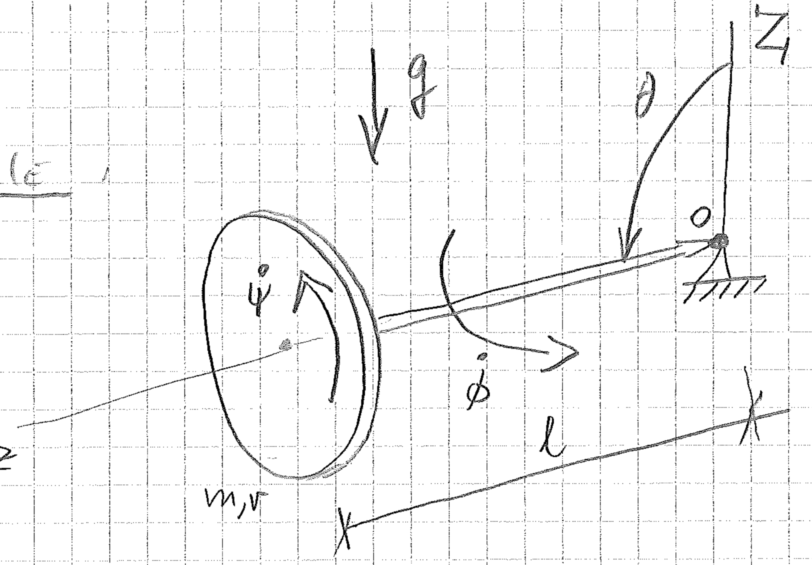



Steady precession for $\theta = 90°$ ?

Steady precession for $\theta = 60°$ ?

[redacted]

Example :




a.) Steady motion for $\theta = 90°$ ?
b.) Steady motion for $\theta = 60°$ ?

a.) $$T = \frac{1}{2}\left[ I\left(\dot\phi^2 \sin^2\theta + \dot\theta^2\right) + I_s\left(\dot\phi\cos\theta + \dot\psi\right)^2 \right]$$

$$V = mgl\cos\theta$$

$$L = T - V.$$

$\phi$ and $\psi$ are ignorable coordinates.
$\theta$ is a nonignorable coordinate.

Steady motion is formulated in relation to the equation of motion of the nonignorable coordinate:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = I\ddot\theta$$

$$\frac{\partial L}{\partial \theta} = I\dot\phi^2\sin\theta\cos\theta + I_s\left(\dot\phi\cos\theta + \dot\psi\right)\cdot(-\sin\theta)\dot\phi$$
$$+ mgl\sin\theta$$

Eq. of motion:

$$I\ddot{\theta} - \sin\theta\left(I\dot{\phi}^2\cos\theta - I_s(\dot{\phi}\cos\theta + \dot{\psi})\dot{\phi} + mgl\right) = 0 \qquad (\mathrm{I})$$

Steady motion: $\Rightarrow \dot{\theta} = \ddot{\theta} = 0$.

For $\theta = 90°$ we obtain

$$0 = \underbrace{\sin 90°}_{=1}\left(I\dot{\phi}^2\underbrace{\cos 90°}_{=0} - I_s(\dot{\phi}\underbrace{\cos 90°}_{=0} + \dot{\psi})\dot{\phi} + mgL\right)$$

$$\Rightarrow \quad I_s\dot{\psi}\dot{\phi} = mgl$$

With $I_s = \frac{1}{2}mr^2$, this leads to

$$\boxed{\dot{\phi} = \frac{2gl}{\dot{\psi}r^2}}$$

b) For $\theta = 60°$, Eq. (I) becomes:

$$0 = \sin 60°\left(I\dot{\phi}^2\underbrace{\cos 60°}_{=\frac{1}{2}} - I_s(\dot{\phi}\underbrace{\cos 60°}_{=\frac{1}{2}} + \dot{\psi})\dot{\phi} + mgl\right)$$

$$\Rightarrow I\dot{\phi}^2\cdot\frac{1}{2} - I_s(\dot{\phi}\cdot\frac{1}{2} + \dot{\psi})\dot{\phi} + mgl = 0$$

$$\frac{1}{2}(I - I_s)\dot{\phi}^2 - I_s\dot{\psi}\dot{\phi} + mgl = 0$$

Inserting $I_s = \frac{1}{2}mr^2$ and $I = \frac{1}{4}mr^2 + ml^2$ (Steiner's rule) this results in

$$\boxed{\dot{\phi} = \frac{2r^2\dot{\psi} \pm 2\sqrt{r^4\dot{\psi}^2 - (4l^2 - r^2)\,2gl}}{4l^2 - r^2}}$$

In addition, the quantity under the root must be non-negative:

$$r^4 \dot{\varphi}^2 - (4l^2 - r^2)\, 2gl \geq 0$$

$$\boxed{\dot{\varphi}^2 \geq \left(\frac{4l^2 - r^2}{r^4}\right) 2gl}$$

which is a condition on the minimally required spin when $\theta = 60°$,