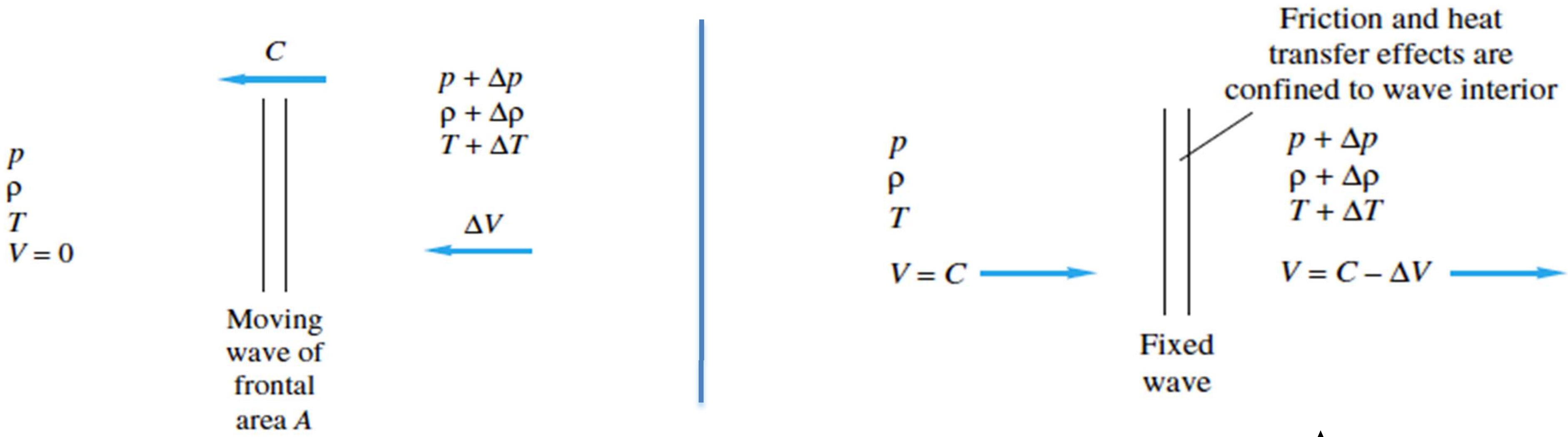
Fluid mechanics (wb1225)

Lecture 12: compressible flow

Example

[1]

Pressure wave



conservation of mass (continuity): $\rho AC = (\rho + \Delta\rho)A(C - \Delta V) \;\Rightarrow\; \Delta V = C\dfrac{\Delta\rho}{\rho + \Delta\rho}$

momentum balance:

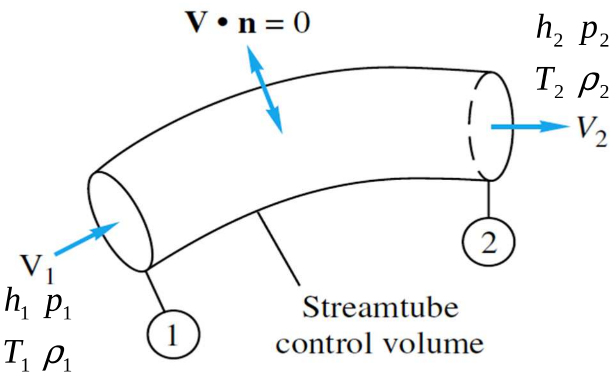
$F = \dot{m}\,(V_{out} - V_{in}) \;\Rightarrow\; pA - (p + \Delta p)A = (\rho AC)(C - \Delta V - C) \;\Rightarrow\; \Delta p = \rho C \Delta V$

combine:

$$C^2 = \frac{\Delta p}{\Delta\rho}\left(1 + \frac{\Delta\rho}{\rho}\right) \xrightarrow{\Delta\rho/\rho \to 0} a^2 = \left.\frac{\partial p}{\partial\rho}\right|_S = kRT \quad \begin{cases} p = \rho RT & \text{ideal gas} \\ p\rho^{-k} = \text{const} & \text{adiab. proc.} \end{cases}$$

Adiabatic gas flow

streamtube



Conservation of energy for stationary flow under adiabatic conditions and without technical work:

- no heat transfer
- effect of viscosity occurs where V = 0
  (no technical work)

ideal gas

$$h = c_p T$$

$$h_1 + \tfrac{1}{2}V_1^2 + \cancel{gz_1} = h_2 + \tfrac{1}{2}V_2^2 + \cancel{gz_2}$$

changes in potential energy are negligible

$$c_p T + \tfrac{1}{2}V^2 = \text{constant}$$

also valid in case of losses

$$\left.\begin{aligned} c_p T + \tfrac{1}{2}V^2 &= c_p T_0 \\ a^2 = \kappa R T &= (\kappa - 1) c_p T \end{aligned}\right\} \Rightarrow \boxed{1 + \frac{\kappa - 1}{2}\mathrm{Ma}^2 = \frac{T_0}{T}}$$

$$\mathrm{Ma} = \frac{V}{a}$$

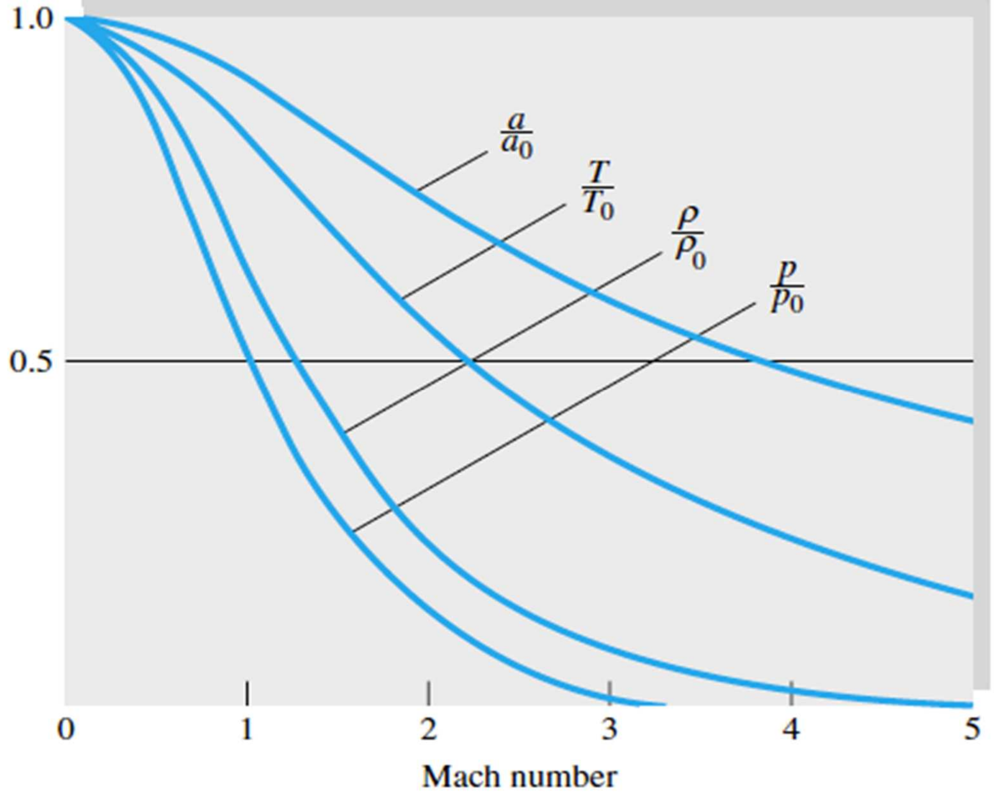
Adiabatic gas flow

$$\frac{T_0}{T} = 1 + \frac{\kappa - 1}{2}\mathrm{Ma}^2$$

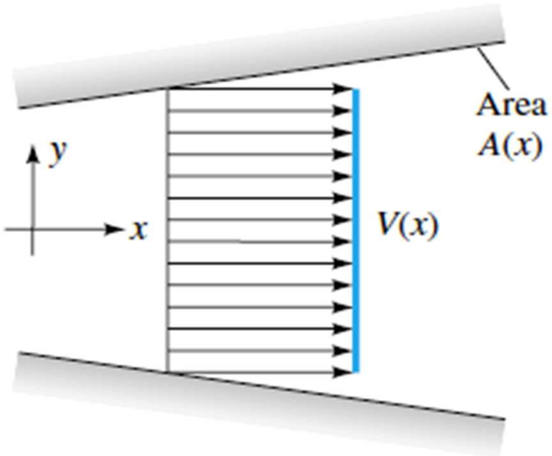
$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\kappa}{\kappa-1}} = \left[1 + \frac{\kappa - 1}{2}\mathrm{Ma}^2\right]^{\frac{\kappa}{\kappa-1}}$$

$$\frac{\rho_0}{\rho} = \left(\frac{T_0}{T}\right)^{\frac{1}{\kappa-1}} = \left[1 + \frac{\kappa - 1}{2}\mathrm{Ma}^2\right]^{\frac{1}{\kappa-1}}$$

$\kappa = 1.4$



One-dimensional isentropic flow



continuity:

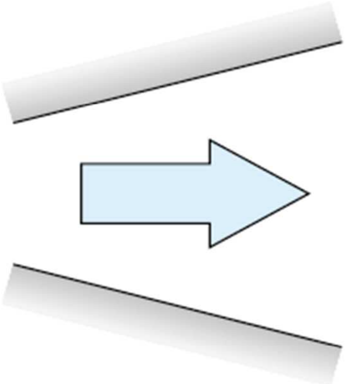
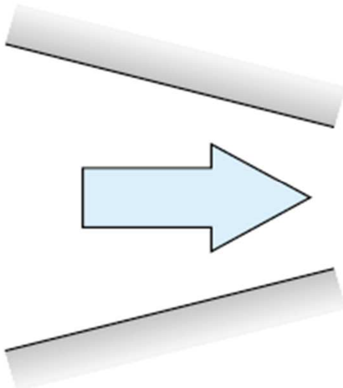
$$\dot{m} = \rho(x)V(x)A(x) = \text{const.} \quad \Rightarrow \quad \frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

momentum: $\dfrac{dp}{\rho} + VdV = 0$

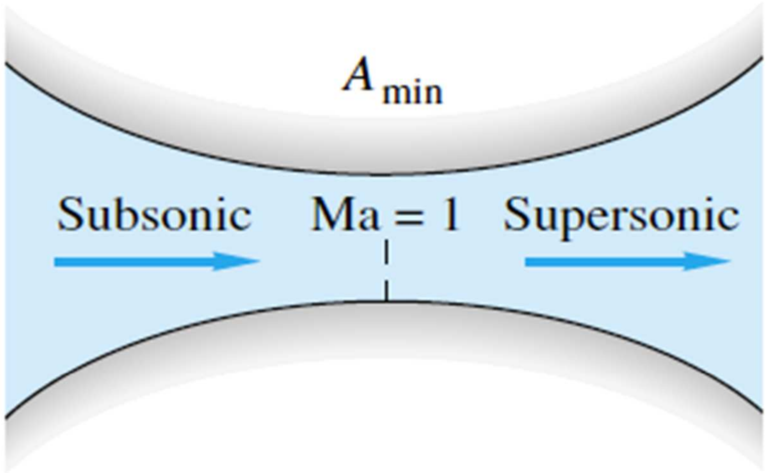
sound: $dp = a^2 d\rho$

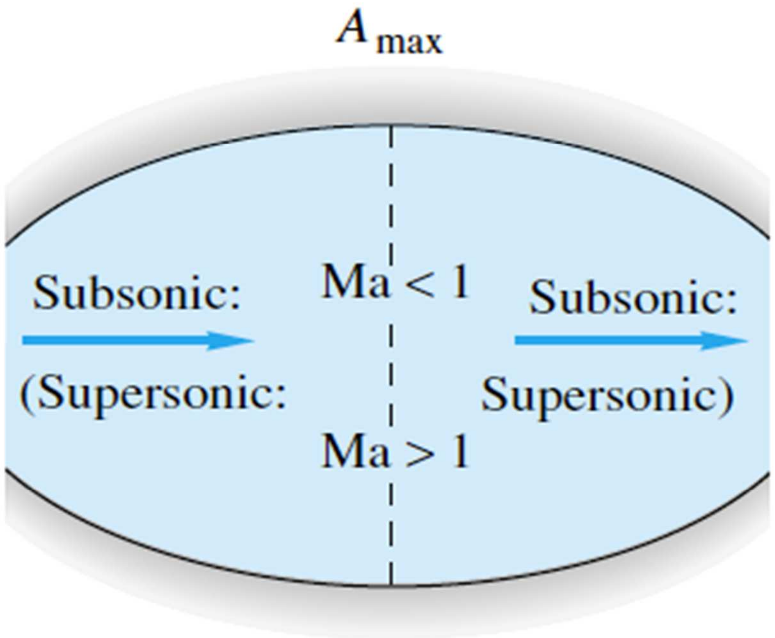
$$\boxed{\frac{dV}{V} = \frac{dA}{A}\frac{1}{\mathrm{Ma}^2 - 1} = -\frac{dp}{\rho V^2}}$$

One-dimensional isentropic flow

Duct geometry		*Subsonic* Ma < 1	*Supersonic* Ma > 1
	$dA > 0$	$dV < 0$ $dp > 0$ Subsonic diffuser	$dV > 0$ $dp < 0$ Supersonic nozzle
	$dA < 0$	$dV > 0$ $dp < 0$ Subsonic nozzle	$dV < 0$ $dp > 0$ Supersonic diffuser

Sonic flow through a nozzle





Ideal gas flow

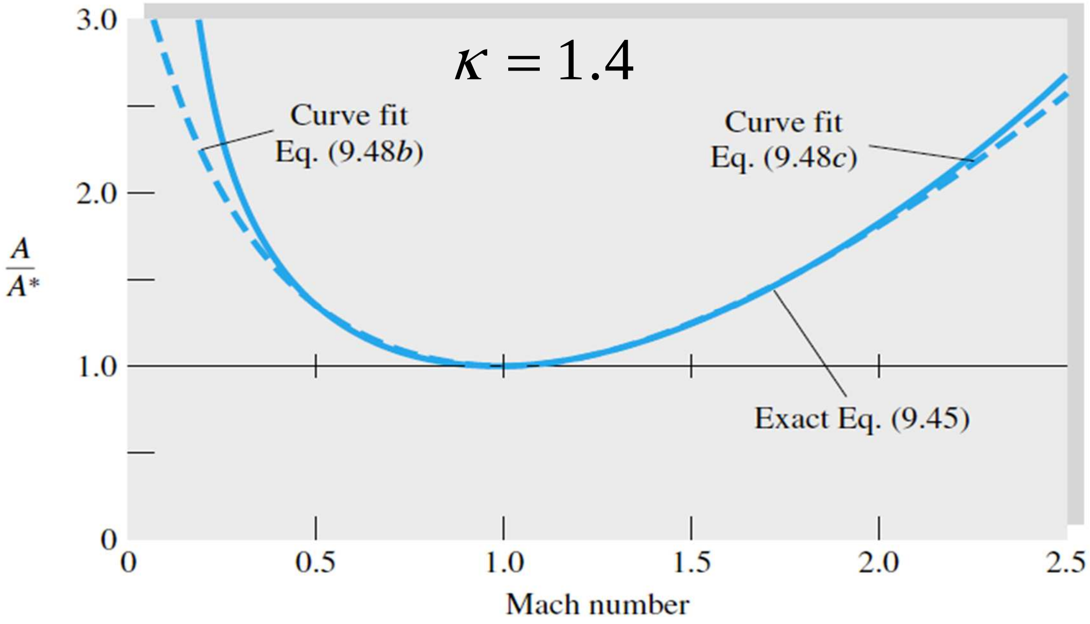
$$\rho V A = \rho^* V^* A^* \quad \text{(sonic conditions, Ma = 1)} \quad \Rightarrow \frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V}$$

$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho} = \frac{\left[1+\frac{\kappa-1}{2}\mathrm{Ma}^2\right]^{\frac{1}{\kappa-1}}}{\left[1+\frac{\kappa-1}{2}\cdot 1^2\right]^{\frac{1}{\kappa-1}}} = \left\{\frac{2}{\kappa+1}\left[1+\frac{\kappa-1}{2}\mathrm{Ma}^2\right]\right\}^{\frac{1}{\kappa-1}}$$

$$\frac{V^*}{V} = \frac{(\kappa R T^*)^{1/2}}{V} = \frac{(\kappa R T)^{1/2}}{V}\left(\frac{T^*}{T}\right)^{\frac{1}{2}}\left(\frac{T_0}{T}\right)^{\frac{1}{2}} = \frac{1}{\mathrm{Ma}}\left\{\frac{2}{\kappa+1}\left[1+\frac{\kappa-1}{2}\mathrm{Ma}^2\right]\right\}^{\frac{1}{2}}$$

$$\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\frac{1+\frac{1}{2}(\kappa-1)\mathrm{Ma}^2}{\frac{1}{2}(\kappa+1)}\right]^{\frac{1}{2}\frac{\kappa+1}{\kappa-1}}$$

Choking



$$\frac{A}{A^*} = \frac{1}{\mathrm{Ma}}\left[\frac{1+\frac{1}{2}(\kappa-1)\mathrm{Ma}^2}{\frac{1}{2}(\kappa+1)}\right]^{\frac{1}{2}\frac{\kappa+1}{\kappa-1}}$$

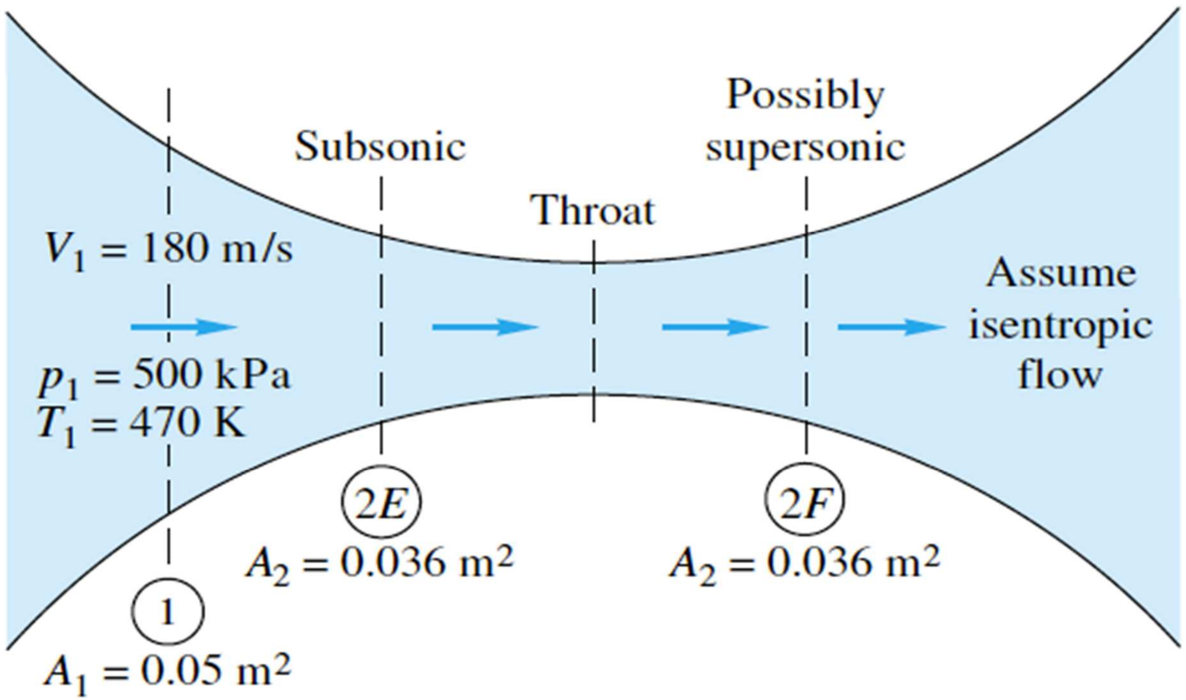
$$\dot{m}_{\max} = \rho^* A^* V^* = \rho_0\left(\frac{2}{\kappa-1}\right)^{\frac{1}{\kappa-1}} A^*\left(\frac{2\kappa}{\kappa+1}RT_0\right)^{\frac{1}{2}}$$

$$= \kappa^{\frac{1}{2}}\left(\frac{2}{\kappa+1}\right)^{\frac{1}{2}\frac{\kappa+1}{\kappa-1}} A^*\rho_0(RT_0)^{\frac{1}{2}}$$

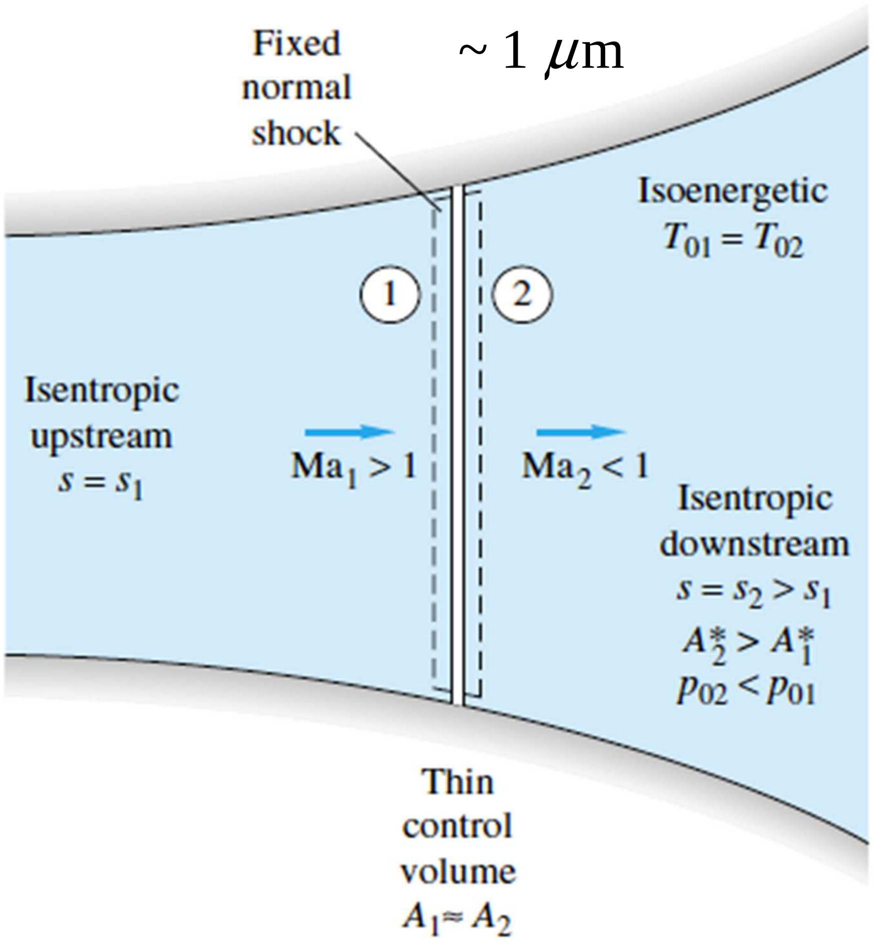
$$\xrightarrow{\kappa=1.4} \dot{m}_{\max} = \frac{0.6847\, p_0 A^*}{(RT_0)^{1/2}}$$

Example 9.4

Air flows isentropically through a duct. At (1) the area is 0.05 m^2 and V_1 = 180 m/s, p_1 = 500 kPa, and T_1 = 470 K. Compute (a) T_0, (b) Ma_1, (c) p_0, and (d) both A^* and $\dot{m}$. If at (2) the area is 0.036 m^2, compute Ma_2 and p_2 for (e) subsonic and (f) super-sonic flow. Assume κ = 1.4.



Normal shock wave



continuity:

$$A_1 \approx A_2$$

$$\rho_1 V_1 = \rho_2 V_2 = \text{const.}$$

momentum:

$$p_1 - p_2 = \rho_2 V_2^2 - \rho_1 V_1^2$$

energy:

$$h_1 + \tfrac{1}{2}\rho_1 V_1^2 = h_2 + \tfrac{1}{2}\rho_2 V_2^2 = h_0 = \text{const.}$$

ideal gas law:

$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \qquad h = c_p T \qquad \kappa = \text{const.}$$

Rankine-Hugeniot relations

eliminate V_1 and V_2 :

$$h_2 - h_1 = \tfrac{1}{2}(p_2 - p_1)\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)$$

introduce ideal gas law:

$$\frac{\rho_2}{\rho_1} = \frac{1+\beta\, p_2/p_1}{\beta + p_2/p_1}, \quad \beta = \frac{\kappa+1}{\kappa-1}$$

entropy change accross shock:

$$s_2 - s_1 = c_v \ln\left[\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\kappa}\right]$$

2nd Law of Thermodynamics:

$$s_2 \geq s_1 \quad \Rightarrow \quad \begin{cases} p_2 \geq p_1 \\ \rho_2 \geq \rho_1 \end{cases}$$

isentropic flow (Poisson relation):

$$\frac{\rho_2}{\rho_1} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{\kappa}}$$

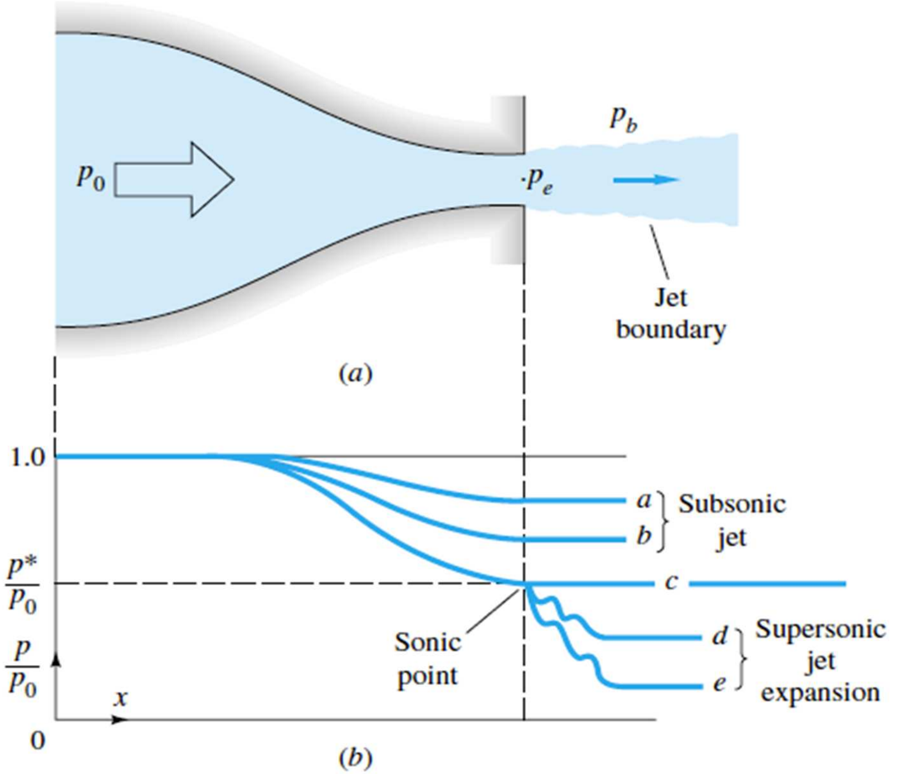
$\frac{p_2}{p_1}$	ρ_2/ρ_1 Eq. (9.51)	ρ_2/ρ_1 Isentropic	$\frac{s_2 - s_1}{c_v}$
0.5	0.6154	0.6095	−0.0134
0.9	0.9275	0.9275	−0.00005
1.0	1.0	1.0	0.0
1.1	1.00704	1.00705	0.00004
1.5	1.3333	1.3359	0.0027
2.0	1.6250	1.6407	0.0134

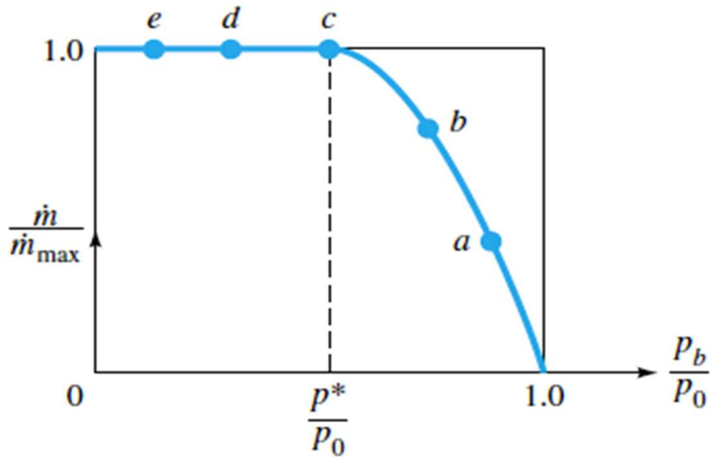
Summary

- $Ma_1>1$, $Ma_2<1$
- $\rho_2 > \rho_1$
- $s_2 > s_1$, $A_2^* > A_1^*$
- weak shocks are nearly isentropic

Example 9.8

A converging nozzle has a throat area of 6 cm^2 and stagnation air conditions of 120 kPa and 400 K. Compute the exit pressure and mass flow if the back pressure is (a) 90 kPa and (b) 45 kPa. Assume κ = 1.4.





Source

1. An F/A-18 Hornet photographed just as it broke the sound barrier, photo courtesy of Ensign John Gay, USS Constellation, US Navy,

The rest of the pictures are from the book of Frank M. White, *Fluid Mechanics*, McGraw-Hill Series in Mechanical Engineering.