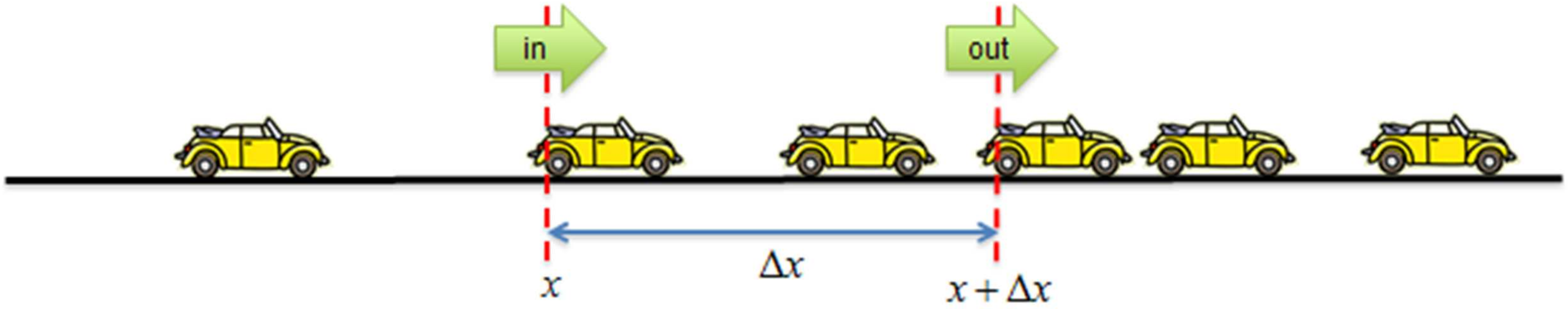
Fluid mechanics (wb1225)

Lecture 6: conservation laws in differential forms

Traffic waves



ρ : density [number of cars / 100 m]

U : velocity [m/s]

[change in #cars in section Δx] = [# of cars that enters in Δt] − [# of cars that leaves in Δt]

$$\Delta M \quad = \quad \rho(x)U(x)\cdot\Delta t \; - \; \rho(x+\Delta x)U(x+\Delta x)\cdot\Delta t$$

$$\frac{\Delta M}{\Delta x} = \frac{\rho(x)U(x) - \rho(x+\Delta x)U(x+\Delta x)}{\Delta x}\cdot\Delta t$$

$$\Delta\rho = \frac{\Delta M}{\Delta x} \qquad \frac{\Delta\rho}{\Delta t} = \frac{\rho(x)U(x) - \rho(x+\Delta x)U(x+\Delta x)}{\Delta x} \Rightarrow \lim_{\substack{\Delta x\to 0\\ \Delta t\to 0}} \Rightarrow \boxed{\frac{\partial\rho}{\partial t} = -\frac{\partial(\rho U)}{\partial x}}$$

Traffic waves

Conservation of Mass:

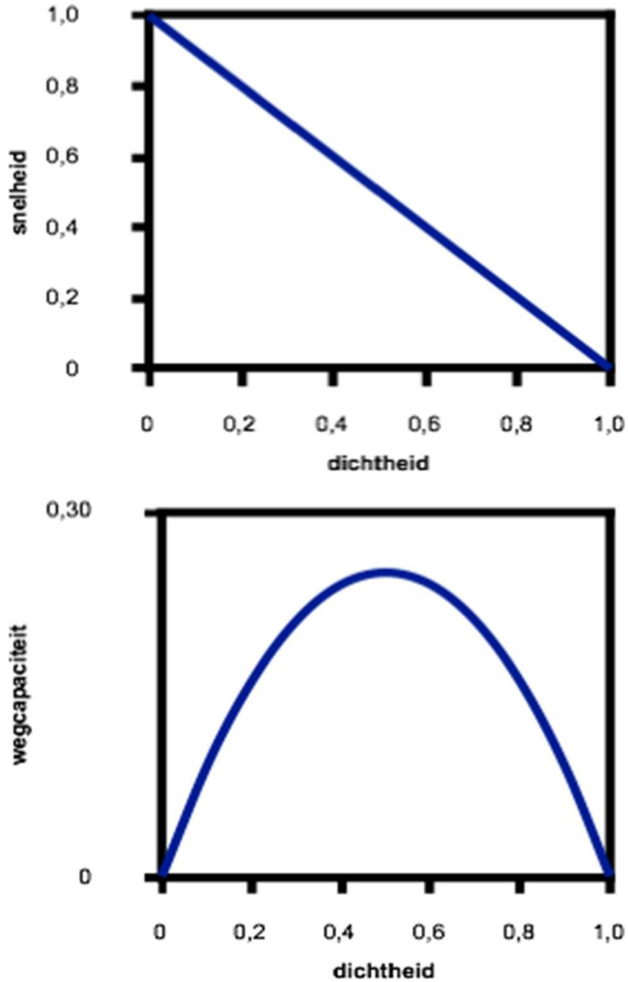
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

driver behavior:

$$u = u_0 \left(1 - \rho / \rho_0\right)$$

equation of motion:

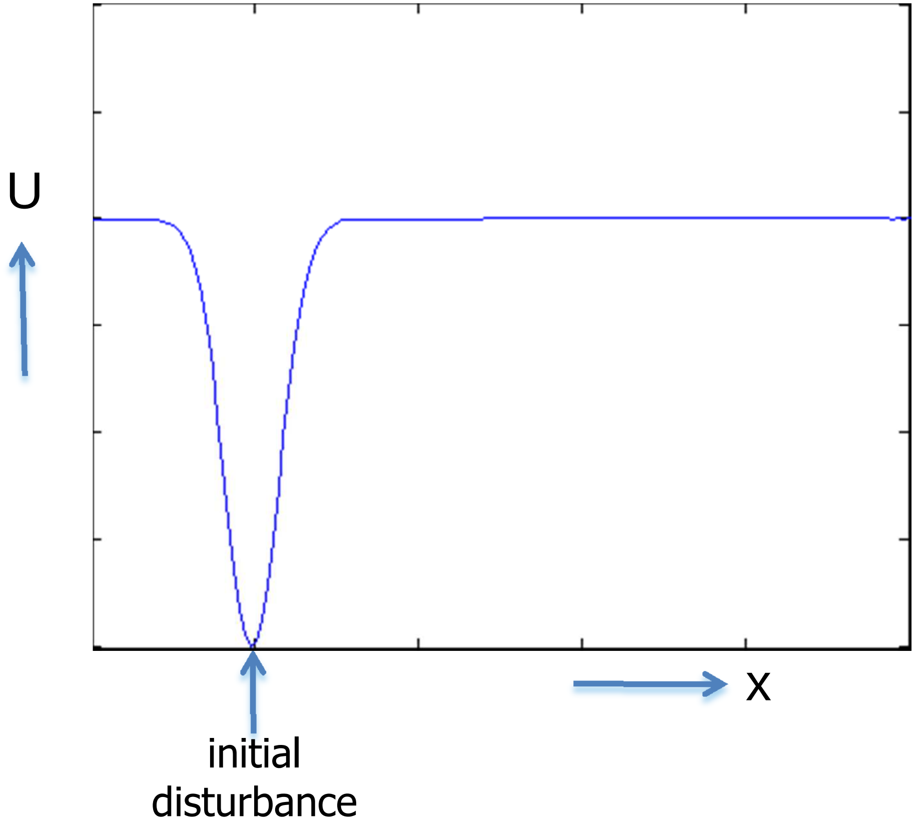
$$\frac{\partial u}{\partial t} + (2u - u_0)\frac{\partial u}{\partial x} = 0$$



Traffic wave

← direction of traffic

→ direction of wave



YouTube

[1]

Conservation of mass

$$\underbrace{\int_{CV} \frac{\partial \rho}{\partial t} dV}_{\text{change of mass in CV}} + \underbrace{\sum_i (\rho_i A_i V_i)_{\text{out}}}_{\text{mass flux leaving CV}} - \underbrace{\sum_i (\rho_i A_i V_i)_{\text{in}}}_{\text{mass flux entering CV}} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\rho = \text{constant}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

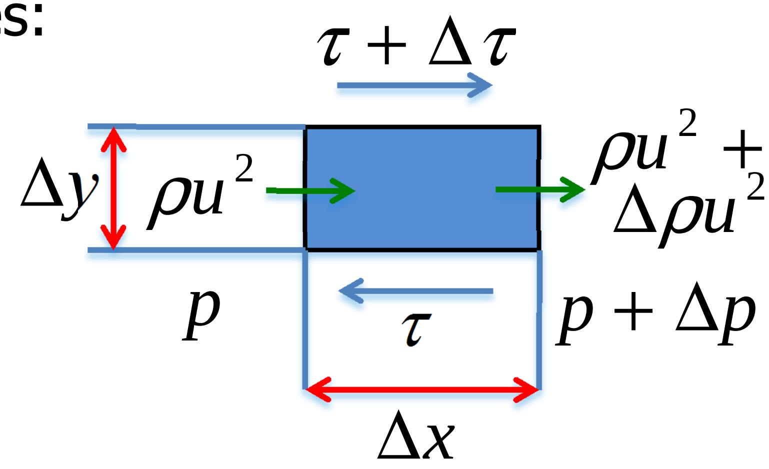
$$0 + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \rightarrow \quad \nabla \cdot \mathbf{u} = 0$$

Conservation of momentum

Differential momentum balance with shear forces:

$$\frac{\Delta(\rho u)}{\Delta t}\Delta x\Delta y=\left[\rho u^2(x,y)-\rho u^2(x+\Delta x,y)\right]\Delta y+$$
$$+\left[p(x,y)-p(x+\Delta x,y)\right]\Delta y+$$
$$+\left[-\tau(x,y)+\tau(x,y+\Delta y)\right]\Delta x$$



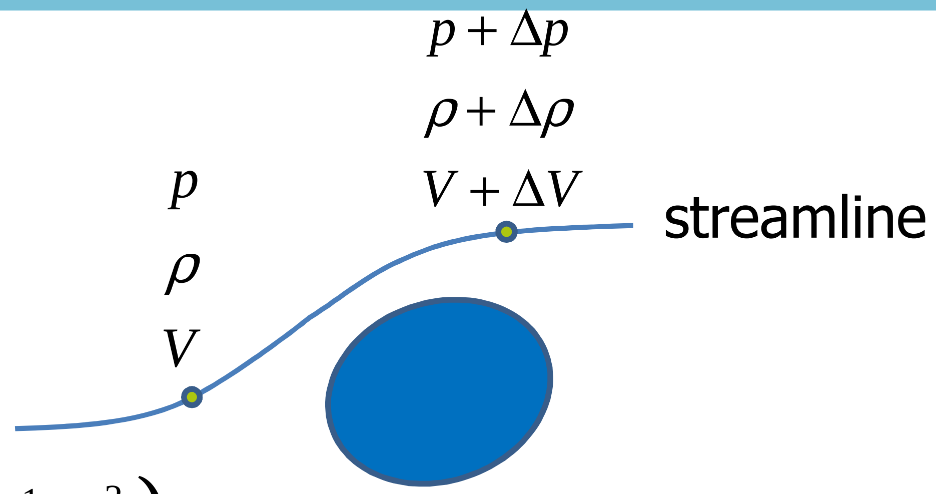
$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial p}{\partial x}+\frac{\partial \tau}{\partial y}$$

Newtonian fluid:

$$\tau=\mu\frac{\partial u}{\partial y}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial z^2}$$

Incompressible flow



Bernoulli:

$$\Delta p + \rho\left(\tfrac{1}{2}(V+\Delta V)^2 - \tfrac{1}{2}V^2\right) = 0$$

$$\Delta p + \rho V \Delta V = 0$$

Speed of sound:

$$a \to \left.\frac{\partial p}{\partial \rho}\right|_S = a^2 \Rightarrow \Delta p = a^2 \Delta\rho$$

$$a^2 \Delta\rho + \rho V \Delta V = 0$$

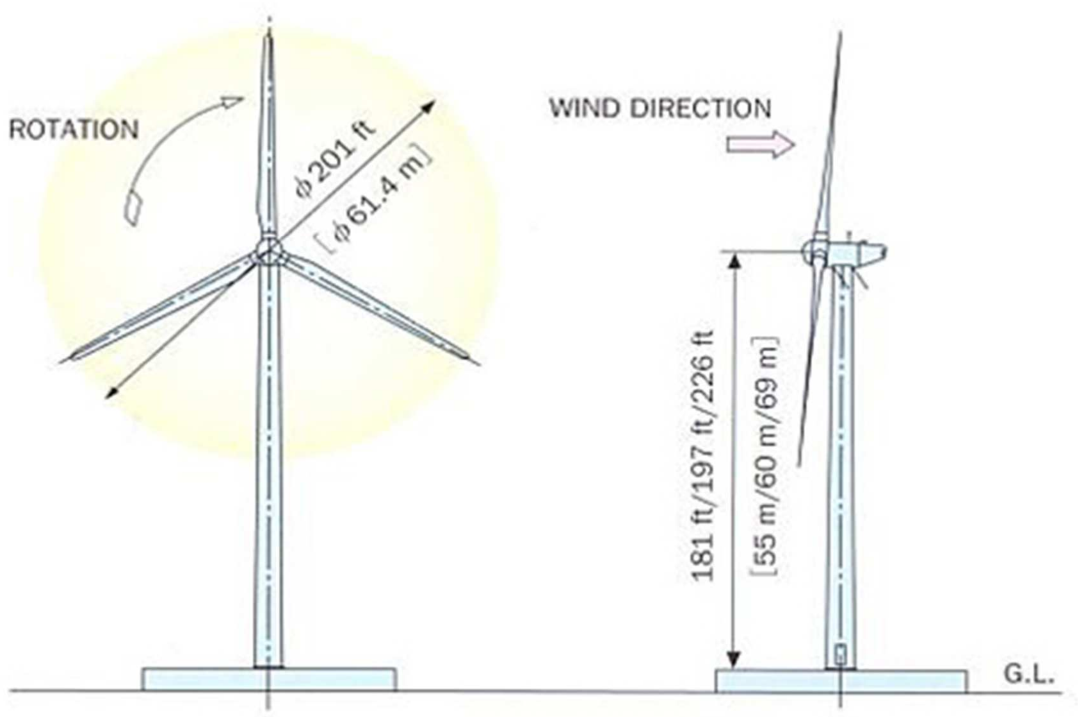
$$\frac{\Delta\rho}{\rho} = \frac{V^2}{a^2}\frac{\Delta V}{V}$$

$$\mathrm{Ma} << 1 \quad \Rightarrow \quad \frac{\Delta\rho}{\rho} << \frac{\Delta V}{V}$$

Flow is effectively incompressible

Example 4.4 (modified)

Compressibility effects for a wind turbine?



Source: www.mhi.co.jp

$$V = \tfrac{1}{2} D\Omega \leq 0.3a = 100 \text{ m/s}$$

$$D = 60 \text{ m}$$

$$\Omega \geq 3.3 \text{ rad/s} = 0.5 \text{ s}^{-1}$$

Oops [2]

Incompressible viscous flow

constant-density Newtonian fluid:

$\rho = \text{constant}$

$\mu = \text{constant}$

stationary flow

incompressible flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial y} = 0$$

$\text{BC}: v = 0 \quad \text{for } y = \pm h: \; v \equiv 0$

NS-equation (momentum equation) for u :

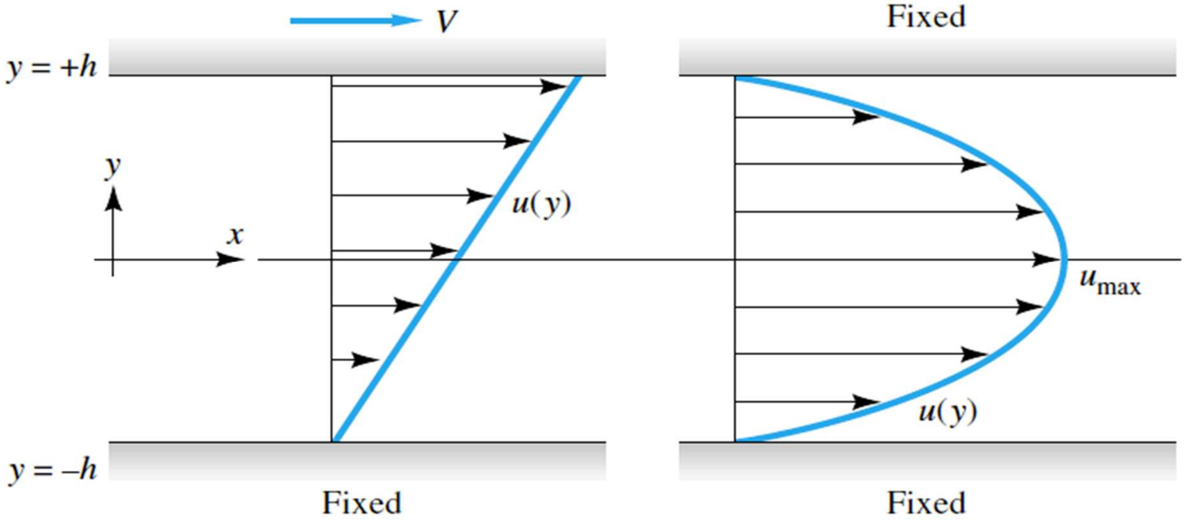
$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

NS-eqn for v : $\frac{\partial p}{\partial y} = 0 \quad \Rightarrow p = p(x) \quad \Rightarrow \frac{\partial p}{\partial x} = \frac{dp}{dx}$

Incompressible viscous flow

$$0 = -\frac{dp}{dx} + \mu \frac{d^2u}{dy^2}$$



Couette flow:

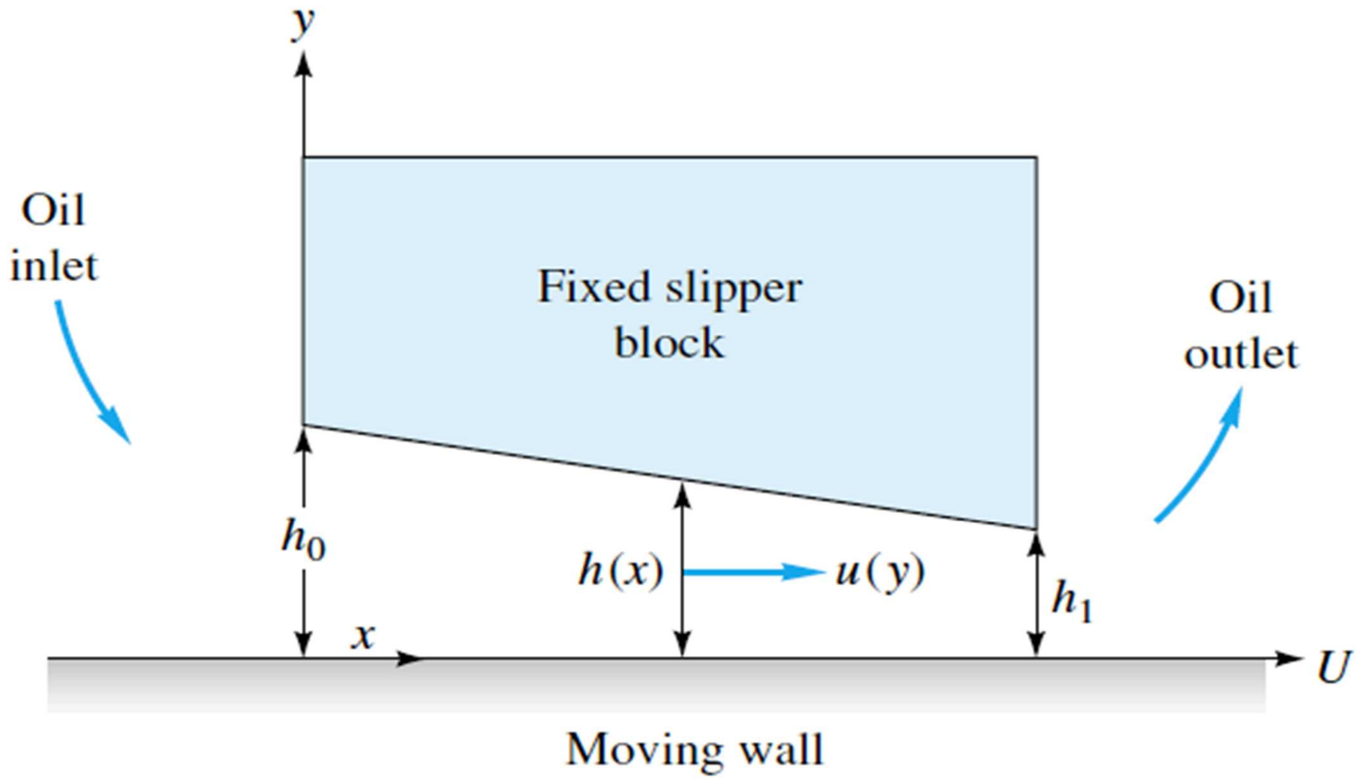
$$\frac{dp}{dx} = 0 \rightarrow \frac{d^2u}{dy^2} = 0 \Rightarrow u = C_1 \cdot y + C_2 \xrightarrow{\text{BC}} \quad u = \tfrac{1}{2}V\left(1 + \frac{y}{h}\right)$$

Poiseuille flow:

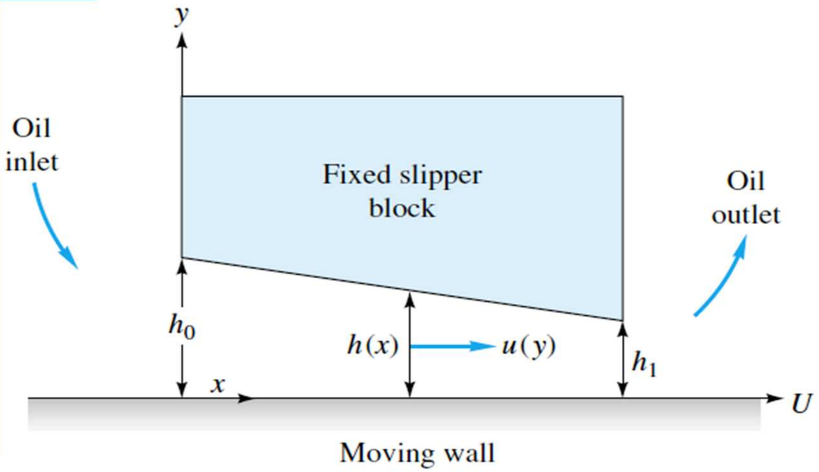
$$\frac{dp}{dx} \neq 0 \rightarrow u = \frac{1}{\mu}\frac{dp}{dx} \cdot \tfrac{1}{2}y^2 + C_1 \cdot y + C_2 \xrightarrow{\text{BC}} \quad u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \frac{y^2}{h^2}\right)$$

Lubrication

P4.83



$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - yh\right) + U\left(1 - \frac{y}{h}\right)$$



$$u = u(y) \quad p = p(x) \quad \rho \frac{\partial u}{\partial t} = 0$$

$$0 = -\frac{dp}{dx} + \mu \frac{d^2 u}{dy^2} \Rightarrow \frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx} = \text{const.}$$

integrate twice:

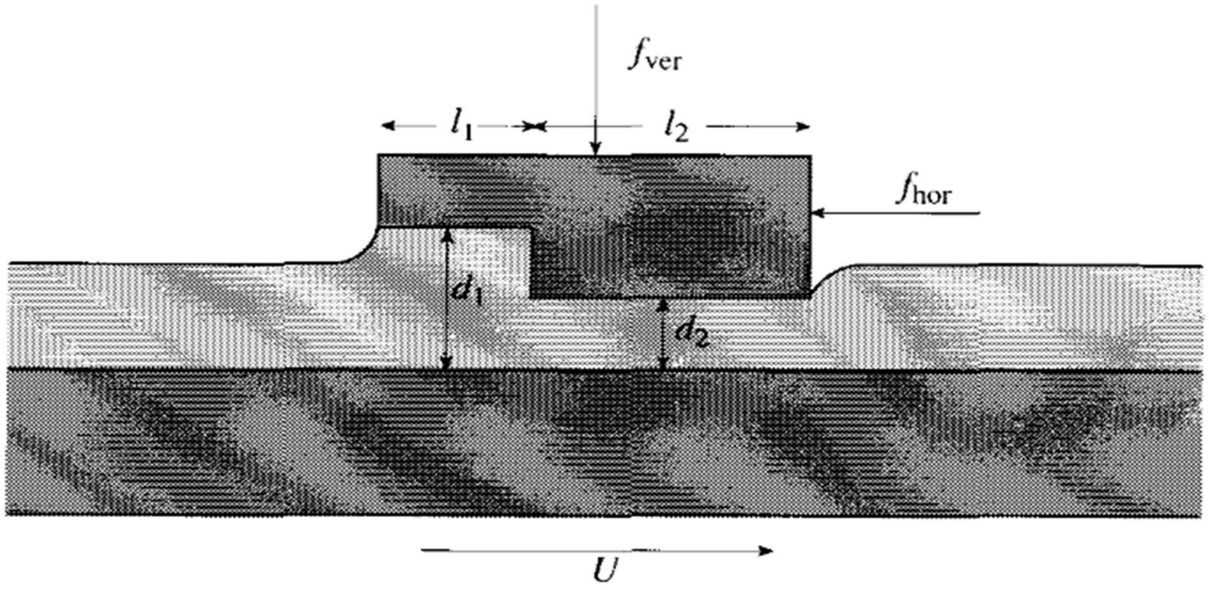
$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

with: $u(0) = U, \quad u(h) = 0$

solution:

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}\left(y^2 - hy\right) + U\left(1 - \frac{y}{h}\right)$$

$$\Rightarrow \quad Q = \frac{1}{2}Uh - \frac{1}{12\mu}\frac{dp}{dx}h^3$$



$$\left.\begin{aligned} Q_1 &= \frac{1}{2}Uh_1 - \frac{1}{12\mu}\frac{p^*}{\ell_1}h_1^3 \\ Q_2 &= \frac{1}{2}Uh_2 + \frac{1}{12\mu}\frac{p^*}{\ell_2}h_2^3 \end{aligned}\right\} \Rightarrow \quad Q_1 = Q_2 \quad \Rightarrow \quad p^* = \frac{6\mu U\left(h_1 - h_2\right)}{\dfrac{h_1^3}{\ell_1} + \dfrac{h_2^3}{\ell_2}}$$

$$f_{\text{ver}} = \frac{1}{2}p^*\left(\ell_1 + \ell_2\right)$$

$$= \frac{3\mu U\left(h_1 - h_2\right)\left(\ell_1 + \ell_2\right)\ell_1\ell_2}{\ell_2 h_1^3 + \ell_1 h_2^3} : \; O\left(\mu U\frac{\ell^3}{\ell h^3}\Delta h\right) = O\left(\mu\frac{U}{h}\ell\frac{\Delta h}{h}\frac{\ell}{h}\right)$$

Lubrication

[3]

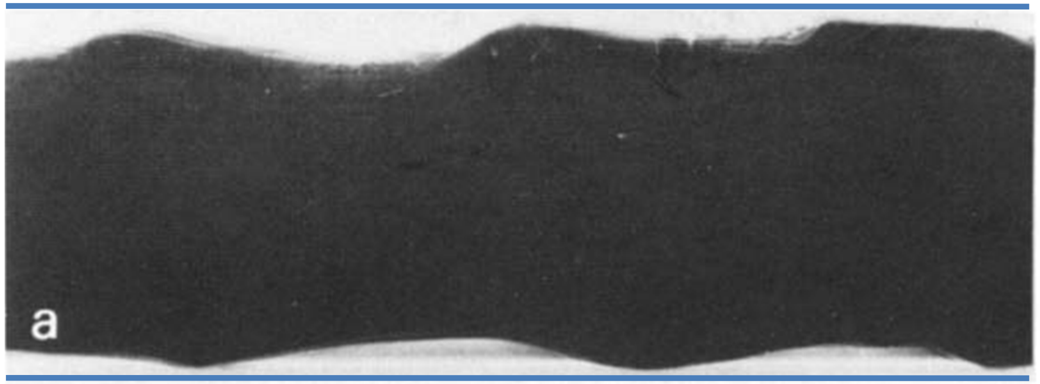
Journal bearing

$$U = \Omega R \quad \alpha = \frac{h}{\pi R}$$

$$F_L \cong \frac{6\mu U}{\alpha^2} \sim \frac{\mu \Omega R^3}{h^2}$$

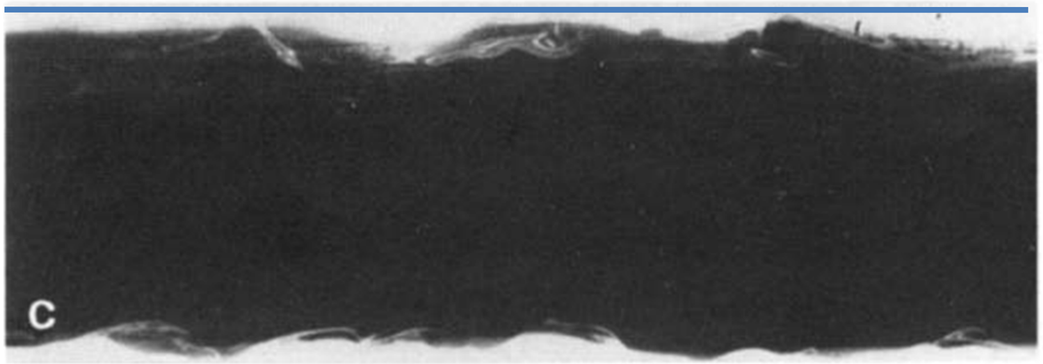
$$F_D \cong \frac{6\mu U}{\alpha} \sim \frac{\mu \Omega R^2}{h}$$

core-annular flow of a 3000 mPa s fuel oil in a 5 cm pipe



0.5 m/s
C_w=0.20

1.0 m/s
C_w=0.03



1.0 m/s
C_w=0.20

Summary

- Chapter 4: 4.2, 4.3, 4.6, 4.11
- Examples: 4.4, 4.6
- Problems:

Source

1. Shockwave traffic jams recreated for first time, *http://youtu.be/Suugn-p5C1M*
2. Het geluid van een windmolen, *http://youtu.be/nf27emmTiCE*
3. Multimedia Fluid Mechanics DVD-ROM, G. M. Homsy, University of California, Santa Barbara