Fluid mechanics (wb1225)

Lecture 7: dimensional analysis

Hollywood

[1]

Buckingham Π theorem

Principle of dimensional homogeneity (PDH):
If an equation truly expresses a proper relationship between variables in a physical process, it will be dimensionally homogeneous; i.e. each of its additive terms will have the same dimensions

$$\left.\begin{array}{ll} n & \text{physical variables} \\ k & \text{fundamental physical quantities} \end{array}\right\} \quad p = n - k \quad \text{dimensionless parameters}$$

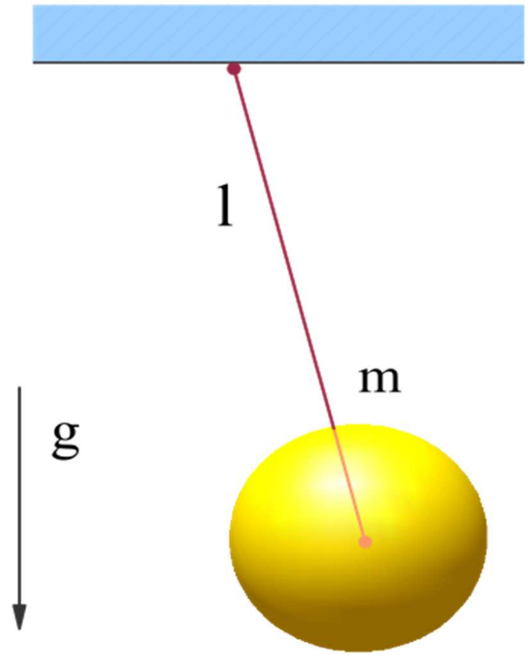
$$f\left(q_1, q_2, \ldots, q_n\right) = 0$$

$$F\left(\pi_1, \pi_2, \ldots, \pi_p\right) = 0$$

$$\pi_i = q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}$$

Example: pendulum

What is the time period t of a pendulum with mass m, length l, and gravity g ?



model: $f(t, m, l, g) = 0$

4 dimensional variables (t, m, l, g),
3 fundamental physical units (T, M, L):
➔ there is 4 – 3 = 1 dimensionless group

$$\Pi = [T]^{\alpha} [M]^{\beta} [L]^{\gamma} \left[L \cdot T^{-2}\right]^{\delta}$$

$$\begin{cases} \alpha - 2\delta = 0 \\ \beta = 0 \\ \gamma + \delta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 2\delta \\ \beta = 0 \\ \gamma = -\delta \end{cases} \qquad \left[\frac{t^2 \cdot g}{l}\right]^{\delta} = 1 \quad \Rightarrow \quad t = K\sqrt{\frac{l}{g}} \quad (K = 2\pi)$$

A more explosive example

$$E = f(\rho, R, t) \Rightarrow E = c \times \frac{\rho R^5}{t^2} \quad (c = 1.033)$$

$$\left[\frac{kg \cdot m^2}{s^2}\right] = \left[\frac{kg}{m^3}\right]^\alpha \cdot [m]^\beta \cdot [s]^\gamma \quad \Rightarrow \quad \alpha = 1 \quad \beta = 5 \quad \gamma = -2$$

t = 0.025 s, R = 140 m

E ~ 90×10^{12} J (90 TJ)

G.I. Taylor (1950)

Scaling of the equations

reference velocity $U \to$ dimensionless velocity $u^* = \dfrac{u}{U}$

reference length $L \to$ dimensionless coordinate $x^* = \dfrac{x}{L}, \ldots$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} - \rho g + \mu \frac{\partial^2 u}{\partial y^2} \qquad \text{with } T = L/U \quad P = \rho a^2$$

$$\frac{U}{T}\frac{\partial u^*}{\partial t^*} + \frac{U^2}{L} u^* \frac{\partial u^*}{\partial x^*} = -\frac{P}{\rho L}\frac{\partial p^*}{\partial x^*} - g + \frac{\mu U}{\rho L^2}\frac{\partial^2 u^*}{\partial (y^*)^2}$$

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} = -\frac{a^2}{U^2}\frac{\partial p^*}{\partial x^*} - \frac{gL}{U^2} + \frac{\mu}{\rho U L}\frac{\partial^2 u^*}{\partial (y^*)^2}$$

Mach: $\mathrm{Ma} = \dfrac{U}{a}$

Froude: $\mathrm{Fr} = \dfrac{U}{\sqrt{gL}}$

Reynolds: $\mathrm{Re} = \dfrac{\rho U L}{\mu}$

Dimensionless numbers

$$\mathrm{Re} = \frac{\rho U L}{\mu} \qquad \frac{\text{inertia forces}}{\text{viscous forces}}$$

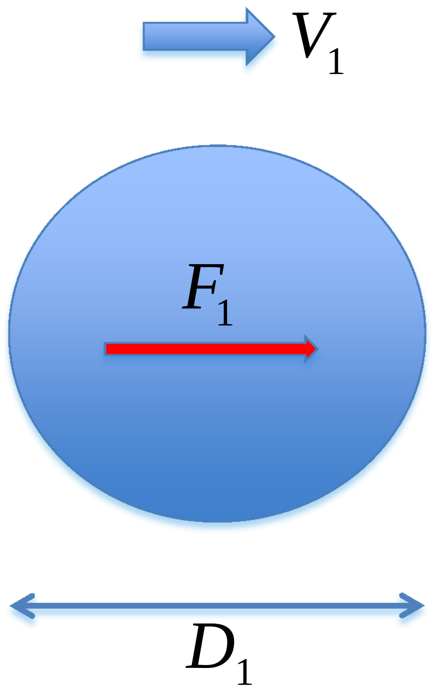
$$\mathrm{Fr} = \frac{U}{\sqrt{gL}} \qquad \frac{\text{inertia forces}}{\text{gravity forces}}$$

$$\mathrm{Ma} = \frac{U}{a} \qquad \frac{\text{flow speed}}{\text{speed of sound}}$$

$$\mathrm{We} = \frac{\rho U^2 L}{\sigma} \qquad \frac{\text{inertia forces}}{\text{surface tension forces}}$$

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A} \qquad \frac{\text{drag force}}{\text{dynamic force}}$$

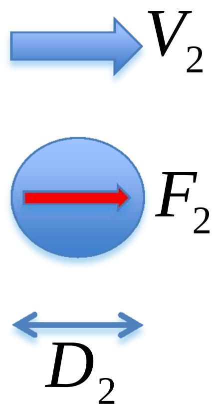
Dynamic similarity



geometric similarity: $x_1^* = x_2^*$

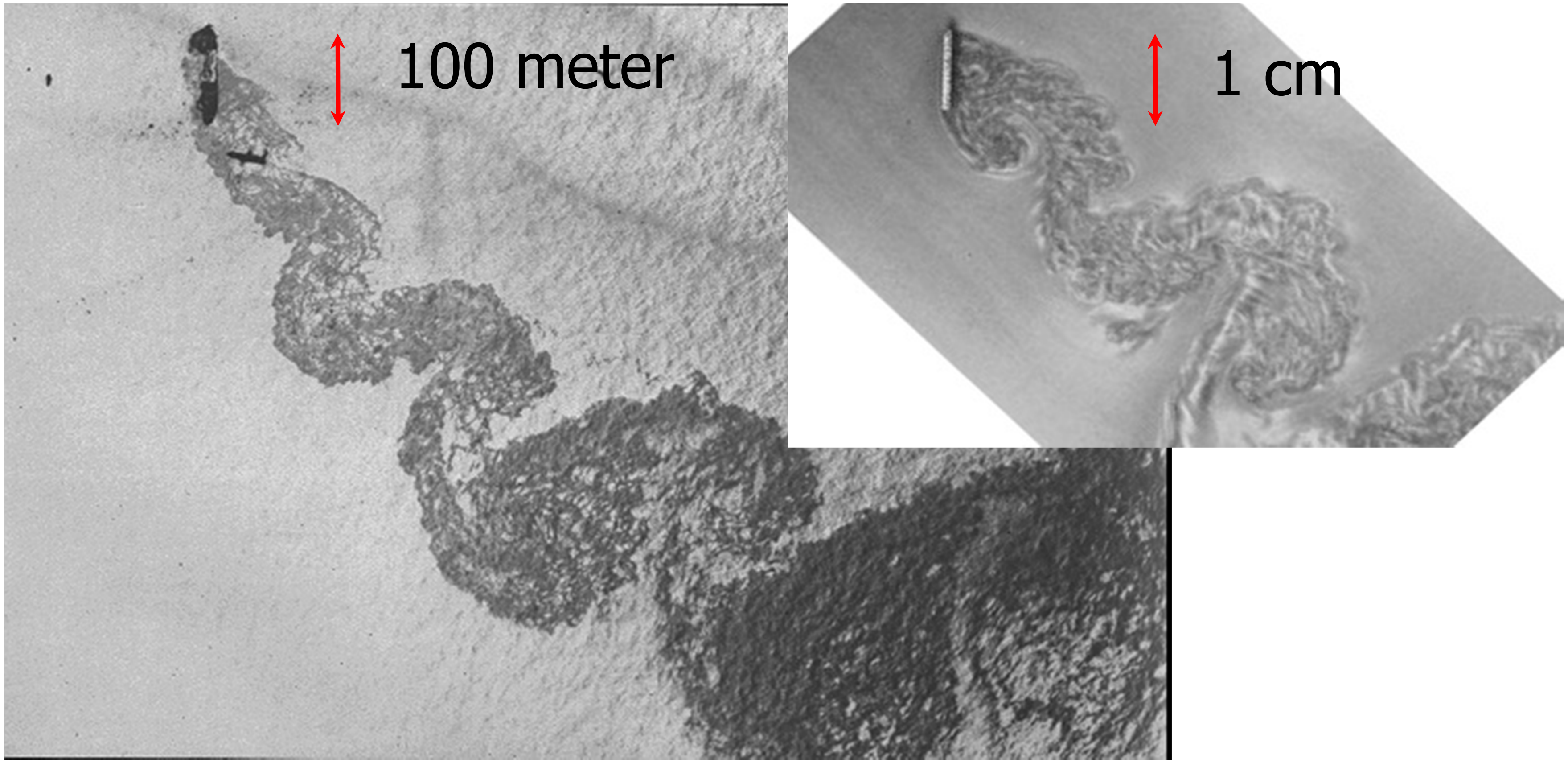
kinematic similarity: $t_1^* = t_2^*$

dynamic similarity: $\mathbf{f}_1^* = \mathbf{f}_2^*$



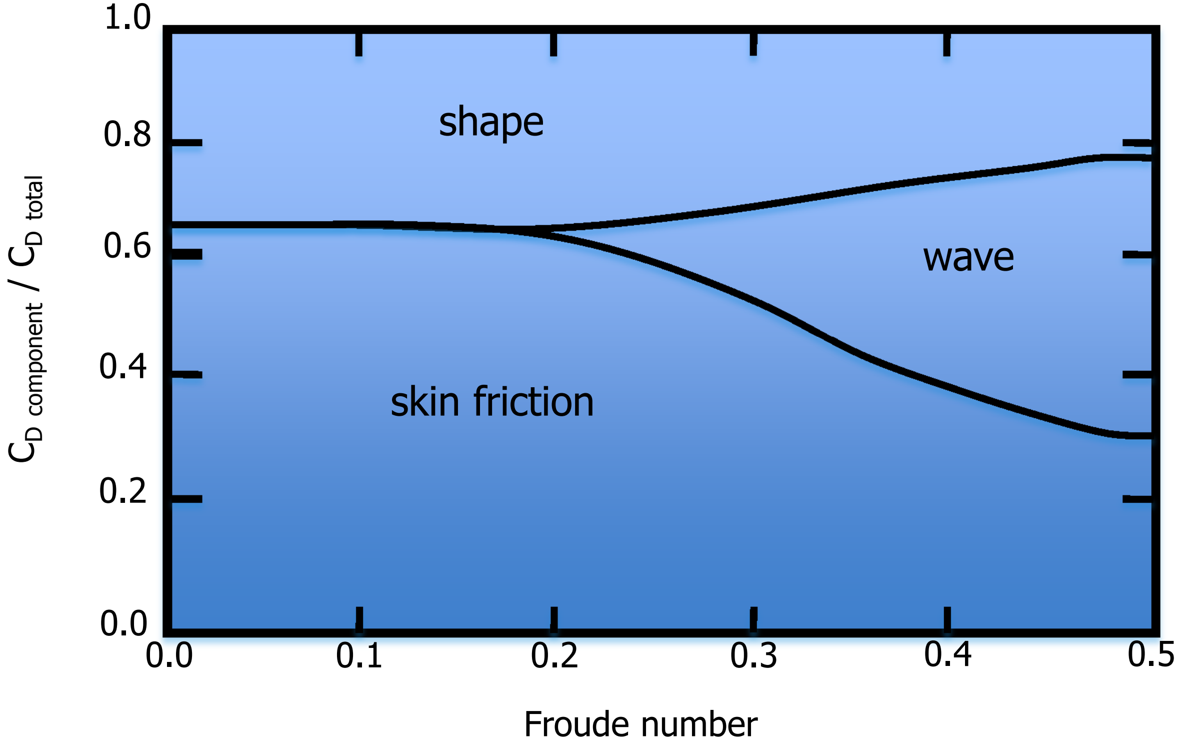
Dynamic similarity requires that all dimensionless numbers are equal
In practice (with many parameters) approximations are needed

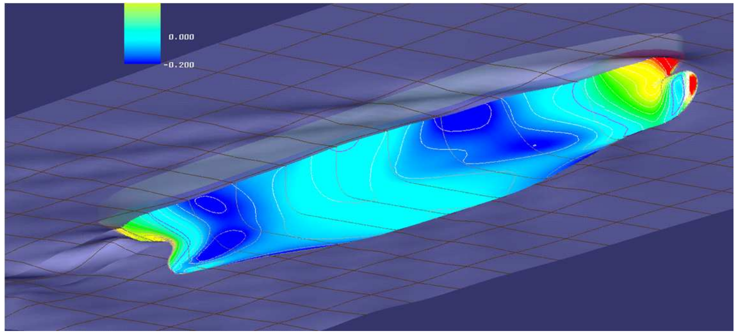
Similarity scaling



Van Dyke, Album of Fluid Motion

Towing tank tests





[2]

[3]

(bow) waves contribute significantly to total drag for Fr > 0.2

Fr is matched in towing tank tests

Re is no longer matched

Summary

- Chapter 5: 5.1-5.5
- Examples: 5.1-5.8
- Problems: see BlackBoard

Source

1. Multimedia Fluid Mechanics DVD-ROM, G. M. Homsy, University of California, Santa Barbara
2. Frank M. White, *Fluid Mechanics*, McGraw-Hill Series in Mechanical Engineering
3. USS George H.W. Bush (CVN 77) , photo by Mr. John Whalen courtesy Northrop Grumman Ship Building