AE4520: Advanced Structural Analysis

Theory of Thin Elastic Plates (Summary)

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Learning Objectives

- Understand the assumptions of thin plate theory
- Derive the expression for the total potential energy of a thin plate- also in the presence of inplane loads
- Derive and Apply the Navier solution of simply supported plates
- Understand the phenomena of plate buckling
- Solve idealised problems using the Rayleigh-Ritz method

Basic Equations

- Strains
  - Bending $\varepsilon_x = -z\, w_{,xx},\ \varepsilon_y = -z\, w_{,yy},\ \gamma_{xy} = -2z\, w_{,xy}$
  - Inplane (moderate rotation)

$$\varepsilon_x = \tfrac{1}{2} w_{,x}^2,\ \varepsilon_y = \tfrac{1}{2} w_{,y}^2,\ \gamma_{xy} = w_{,x} w_{,y}$$

- Moment resultants

$$\begin{bmatrix} M_x \\ M_y \\ M_{yx} \end{bmatrix} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{bmatrix},\quad D = \frac{Et^3}{12(1-\nu^2)}$$

- Stresses

$$\sigma_x = \frac{N_x}{t} + \frac{12M_x}{t^2} z,\ \sigma_y = \frac{N_y}{t} + \frac{12M_y}{t^2} z,\ \tau_{xy} = \frac{N_{xy}}{t} + \frac{12M_{yx}}{t^2} z$$

The Plate Equation

- Equilibrium equations

$$M_{x,x} + M_{yx,y} = Q_x \qquad Q_{x,x} + Q_{y,y} + q = 0$$
$$M_{yx,x} + M_{y,y} = Q_y$$

- Plate Equation $D\left[w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\right] = q$

- Simply-Supported boundary conditions

$$\begin{array}{ll} edge \perp x & edge \perp y \\ w = 0 & w_{,y} = 0 \\ w_{,xx} = 0 & w_{,yy} = 0 \end{array}$$

Navier Solution

- For a SS rectangular plate of dimensions $a \times b$

$$w = \sum_{m=1,n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

- where $$A_{mn} = \frac{a_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}$$

- The Fourier sine coefficients of applied pressure are

$$a_{mn} = \frac{4}{ab}\int_0^b\int_0^a q \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy$$

Navier Solution: inplane loads

- For a SS rectangular plate of dimensions $a \times b$

$$w = \sum_{m=1,n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

- where

$$A_{mn} = \frac{a_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 + N_x\left(\frac{m\pi}{a}\right)^2 + N_y\left(\frac{n\pi}{b}\right)^2}$$

- The Fourier sine coefficients of applied pressure are

$$a_{mn} = \frac{4}{ab}\int_0^b\int_0^a q \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dxdy$$

Total Potential Energy

- Strain energy

$$U = \tfrac{1}{2}\int_0^b\int_0^a D\left[w_{,xx}^2 + w_{,yy}^2 + 2\nu w_{,xx} w_{,yy} + 2(1-\nu) w_{,xy}^2\right] dxdy$$

- Potential energy of inplane loads

$$V_i = \tfrac{1}{2}\int_0^b\int_0^a \left[N_x w_{,x}^2 + N_y w_{,y}^2 + 2N_{xy} w_{,x} w_{,y}\right] dxdy$$

- Potential energy of external pressure

$$V_e = -\int_0^b\int_0^a qw\, dxdy$$

Rayleigh-Ritz method: statics

- Assumed deflection $w = \sum_{i=1}^{N} A_i \phi_i (x, y)$
- The functions ϕ_i satisfy conditions on displacements and rotations
- The mode shape and buckling factor come from:

$$\left(\mathbf{K} - \lambda \mathbf{K}_g\right) \cdot \mathbf{A} = \mathbf{0}$$

- **K** is as before and, the geometric matrix is found from:

$$K_{g-ij} = -\int_0^b \int_0^a N_x \phi_{i,x} \phi_{j,x} + N_y \phi_{i,y} \phi_{j,y} + N_{xy} \left(\phi_{i,x} \phi_{j,y} + \phi_{i,y} \phi_{j,x}\right) dxdy$$

Rayleigh-Ritz method: statics

- Assumed buckling mode $w=\sum_{i=1}^{N} A_i\phi_i(x,y)$
- The functions ϕ_i satisfy conditions on displacements and rotations
- The coefficents are determined from: $\mathbf{K}\cdot\mathbf{A}=\mathbf{f}$

$$K_{ij}=\int_0^b\int_0^a D\begin{bmatrix}\phi_{i,xx}\phi_{j,xx}+\phi_{i,yy}\phi_{j,yy}+\nu\left(\phi_{i,xx}\phi_{j,yy}+\phi_{i,yy}\phi_{j,xx}\right)+\\ +2(1-\nu)\phi_{i,xy}\phi_{j,xy}\end{bmatrix}dxdy$$

- Potential energy of external pressure

$$f_i=\int_0^b\int_0^a q\phi_i\,dxdy$$