4.12 SYMMETRY

Types of symmetry include reflective, skew, axial, and cyclic. If symmetry is recognized and exploited, the size of the FE model is reduced. Thus there is less input data to prepare and less computation to do.

A structure has *reflective* or *mirror* symmetry if there is symmetry of geometry, support conditions, and elastic properties with respect to a plane. Reflective symmetry of structure *and loads* is shown in Fig. 4.12-1a: if reflected by the plane $x = 0$, the left half yields the right half and vice versa. One could say that reflection brings the structure and its loads into "self-coincidence." Analysis of either half yields a complete solution because symmetric loading on a symmetric structure produces symmetric results.

If $P_x = P_y$ in Fig. 4.12-1a, the planes $x = 0$, $y = 0$, $x = y$, and $x = -y$ are all planes of reflective symmetry, and we need analyze only one octant of the structure, using $P_x/2$ as the load. Supports on a symmetry plane in Fig. 4.12-1a must allow only motion radially from the origin $x = y = 0$ (as in Fig. 3.12-2a). A similar example appears in Fig. 4.12-1b: analysis of the right (or left) half of the beam, with rotation θ_z prevented at $x = 0$, provides a complete solution of the problem. These examples are very simple, but one can see that if the structure were large and complicated it would be a waste of effort to ignore symmetry and prepare a model of the entire structure.

Note that loads as well as structure may be cut by a plane of symmetry. In Fig. 4.12-1a, if only half the structure is retained because plane $x = 0$ is used as a plane of reflective symmetry, loads P_y become $P_y/2$ on the half retained. Similarly, if a stiffening beam (as might be used beneath a floor slab) is longitudinally bisected by a plane of reflective symmetry, only half its stiffness is retained.

The problem shown in Fig. 4.12-1c is *antisymmetric* because of the loading. Reflection about the plane $x = 0$, followed by *reversal* of all loads, results in self-coincidence. Again, analysis of half the structure yields a complete solution. Note, however, that support conditions differ in Figs. 4.12-1b and 4.12-1c.

Rules that help in setting the correct support conditions for reflective symmetry are as follows. The conditions stated apply *only* to boundary nodes of the FE model that lie in a plane of reflective symmetry of the entire structure. If the problem is *symmetric:*

1. Translations have no component normal to a plane of symmetry.
2. Rotation vectors have no component parallel to a plane of symmetry.

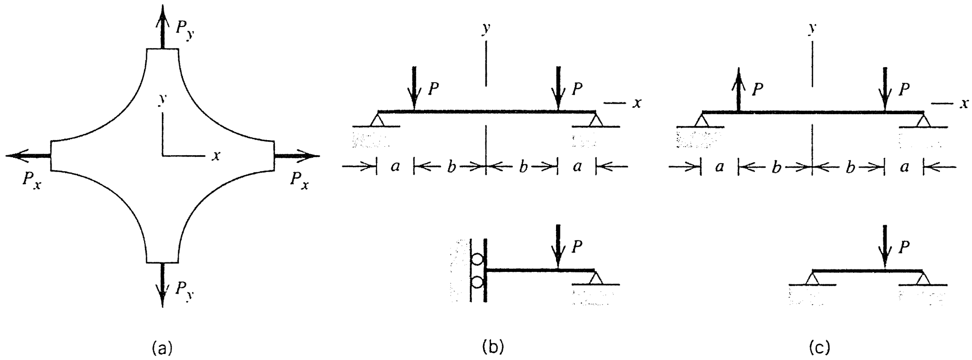

Fig. 4.12-1. (a) Plane structure having reflective symmetry about $x = 0$ and $y = 0$ planes. (b) Beam under symmetric load. (c) Beam under antisymmetric load.

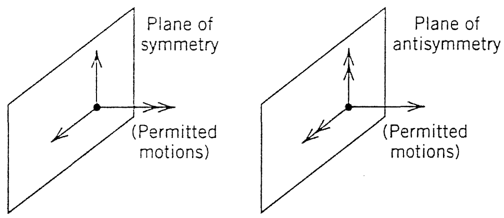

Fig. 4.12-2. The d.o.f. permitted (i.e., not restrained) at a node in a plane of symmetry or antisymmetry. A double-headed arrow represents a rotational d.o.f.

If the problem is *antisymmetric*, that is, symmetric except that loads must be reversed to achieve self-coincidence:

1. Translations have no component parallel to a plane of antisymmetry.
2. Rotation vectors have no component normal to a plane of antisymmetry.

Figure 4.12-2 depicts these rules in terms of d.o.f. *permitted* rather than d.o.f. restrained. The reader should verify that these rules hold for the special cases in Figs. 4.12-1b and 4.12-1c.

If one suspects the presence of symmetries but their nature is not clear, one may do a coarse-mesh analysis, either of the entire structure or a part of it that is obviously treatable by symmetry considerations. Computed results may confirm or refute the existence of the suspected symmetries.

Figure 4.12-3 is an example of how symmetry concepts might be applied even when obvious symmetries are not present [5.4]. By regarding the load as the sum of symmetric and antisymmetric parts, we obtain the cases in Figs. 4.12-3b and 4.12-3c. By superposing solutions of these two cases, we solve the original problem. Thus bending moments in Fig. 4.12-3a are $M_1 = M_4$, $M_2 = M_5 + M_7$, and $M_3 = M_4$. We have traded one solution of the entire structure for two solutions of half the structure. The possible advantage is that

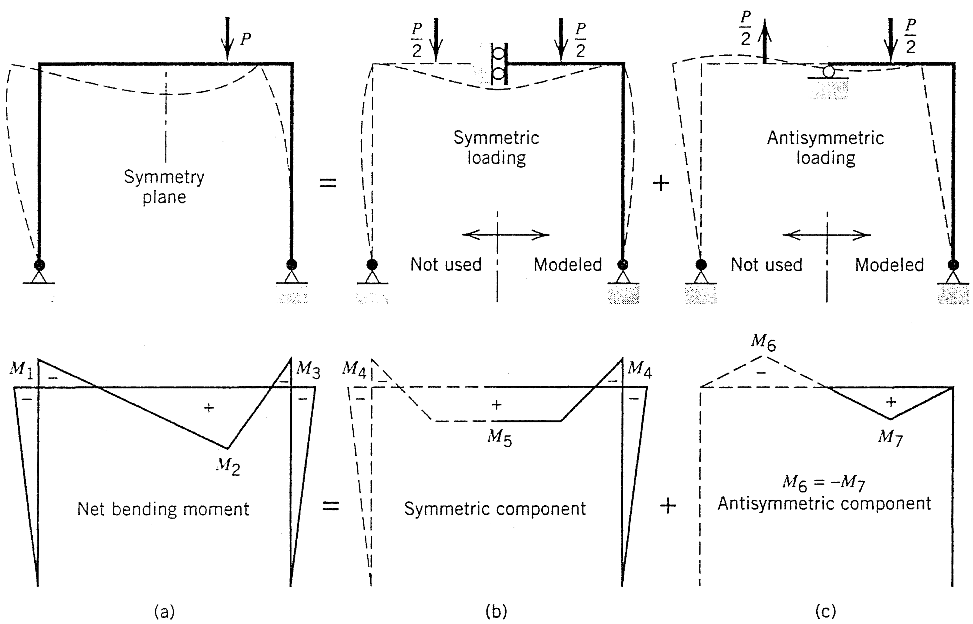

Fig. 4.12-3. Modeling a plane frame problem as the sum of symmetric and antisymmetric cases. (Reproduced from [5.4] by permission of the publisher.)

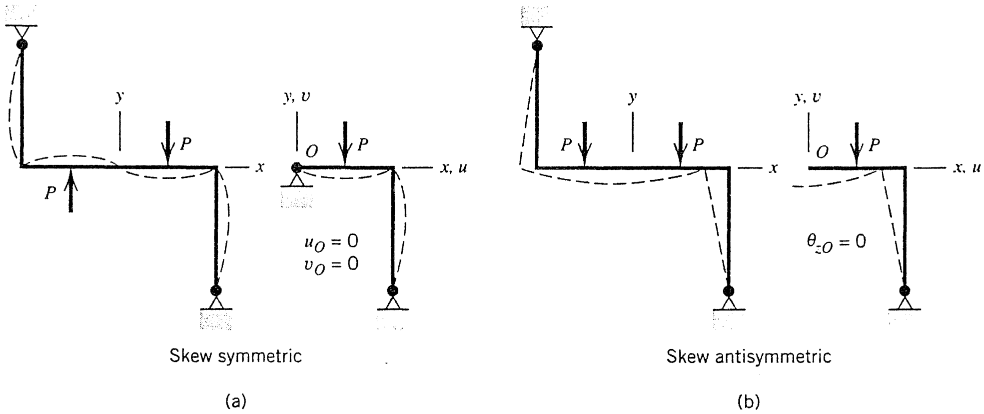

Fig. 4.12-4. Skew symmetry of a plane frame, with loads that are (a) skew symmetric and (b) skew antisymmetric.

the two solutions differ only in loads and support conditions. Such a trade may be advantageous if the structure is geometrically complicated and considerable effort is needed to prepare input data, or if the reduction in number of d.o.f. is important.

Skew or *inversion* symmetry is illustrated in Fig. 4.12-4. In Fig. 4.12-4a, a half-revolution of structure and loads about the z axis (normal to the paper) results in self-coincidence. In Fig. 4.12-4b, a half-revolution followed by reversal of loads results in self-coincidence. In both cases only half the structure need be analyzed, but support conditions at point O are not so readily stated as are support conditions for cases of reflective symmetry [4.7].

Axial symmetry prevails when a solid is generated by rotation of a plane shape about an axis in the plane. Although the structure is three-dimensional, the FE model need be only two-dimensional. Axially symmetric bodies are common and their analysis is discussed separately (Chapter 6).

A structure that is not axially symmetric may yet exhibit a rotational repetition of geometry, material properties, supports, and loads. This circumstance is called *cyclic* symmetry (or sectorial symmetry, or rotational periodicity). An example appears in Fig. 4.12-5a. A complete solution is obtainable by analysis of one repetitive portion, such as that in Fig. 4.12-5b. Other choices of representative repetitive portion are possible. Although only one such portion is needed, it is convenient to speak of "attachment" d.o.f. along AB and CD. Attachment d.o.f. along AB and CD must match exactly—in number, placement, type, and orientation—for the reason that d.o.f. along AB and CD must be constrained to have identical displacements. Specifically, nodes A and C must have the same displacement components in the respective n directions and the same displacement components in the respective s directions. If attachment d.o.f. carry externally applied loads, these loads must be applied on either AB or CD, but not both, as this would apply twice the load intended. In order to exploit cyclic symmetry, it is not necessary that the body be plane or that attachment d.o.f. lie on straight lines. In general, attachment d.o.f. lie on congruent curved surfaces in space, match exactly in position, and use d.o.f. that match in their orientations with respect to these surfaces. Concepts of cyclic symmetry need not be restricted to problems in which repetitions of form and loading appear with rotation about an axis. Similar repetitions may appear in a long slender structure. With appropriate loading, this would be possible in Fig. 4.11-1b, for

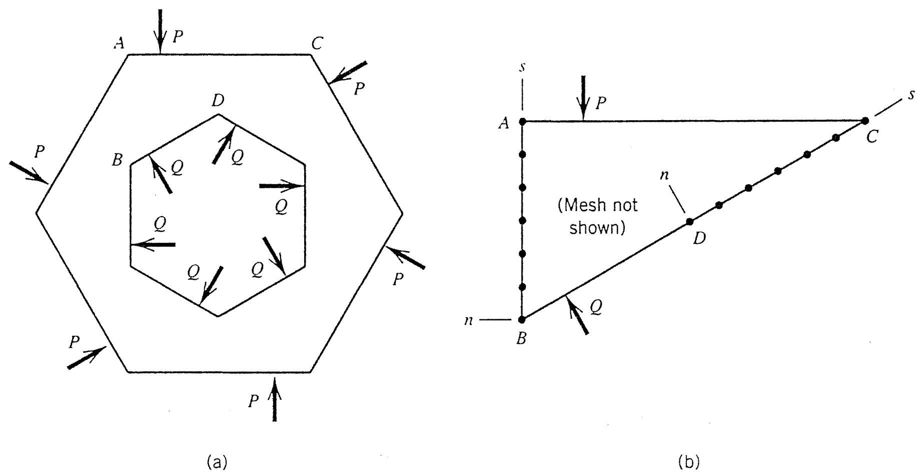

Fig. 4.12-5. (a) Plane structure that exhibits cyclic symmetry. Loads are P and Q. Supports (not shown) exert no force. (b) Typical repeating portion. Nodes on AB and CD are shown but the FE mesh is not shown.

example. This circumstance, less common than cyclic symmetry, may be called "repetitive" symmetry.

Caution. Symmetry concepts should be used sparingly and carefully in problems of vibration and buckling. For example, a uniform, simply supported beam has symmetry about its center but has *anti*symmetric vibration modes as well as symmetric vibration modes. If half the beam were analyzed, the support conditions of Fig. 4.12-1b would permit only symmetric vibration modes, while the support conditions of Fig. 4.12-1c would permit only antisymmetric vibration modes. Similarly, an axisymmetric solid or shell will have many vibration modes that are not axisymmetric. Caution is also needed in static problems that involve nonlinearity because symmetries present when loading begins may subsequently disappear.

4.13 CONSTRAINTS

A *constraint* may merely prescribe the numerical value of a d.o.f. and may then be called a "single-point constraint." The most common example is setting a d.o.f. to zero as a support condition. In the following discussion, "constraint" is used to mean a prescribed relation among d.o.f. (sometimes called a "multipoint constraint"). The problem of Fig. 4.3-3 is an example. In that problem, d.o.f. at nodes 1 and 2 are constrained to follow d.o.f. at nodes 3 and 4 and are replaced by d.o.f. at nodes 3 and 4 prior to assembly of elements. A constraint is roughly the opposite of a release (Section 2.3); however, d.o.f. in a constraint relation need not be physically adjacent.

One way to impose constraints is to use transformation, much as described below Eq. 4.3-2, to eliminate constrained d.o.f. prior to assembly of elements. For each equation of constraint, one d.o.f. can be eliminated. In what follows we describe how constraints may be applied to global equations $\mathbf{KD} = \mathbf{R}$, *after* assembly of elements, to override the elastic relation among d.o.f. to be constrained. We will describe two methods that are used in commercial software: the Lagrange multiplier method, which imposes constraints ex-