AE4520: Advanced Structural Analysis

Total Potential and Strain Energy

Mostafa Abdalla

Learning Objectives

- State and Derive the principle of minimum total potential
- Understand the properties of stress-strain relationships
  - Symmetry of the material stiffness
  - Isotropy
- State and Derive the principle of virtual work and the principle of minimum total potential for plane stress conditions

Summary

- If forces are conservative they can be derived from a scalar potential
- The potential energy of internal forces depends on strain is called strain energy
- Strain energy per unit volume is a material property and defines the stress-strain relation
- For small strain, and zero initial stresses, the strain energy function is quadratic in strains
- For isotropic materials only two independent material constants exist
- For thin-walled structures we can neglect out-of-plane stresses. This leads to a plane stress situation

Stress-Strain relations

- Strain energy in the small strain range
  - **C** is symmetric, $\psi = \frac{1}{2}\boldsymbol{\varepsilon}^t \mathbf{C}\, \boldsymbol{\varepsilon}$
- Stress- strain relation

$$\boldsymbol{\sigma} = \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}} = \mathbf{C}\, \boldsymbol{\varepsilon}$$

- Transformation of material stiffness
  - Stress and strain transformation (depends on rotation of axes)

$$\boldsymbol{\sigma} = \mathbf{T}_\sigma\, \boldsymbol{\sigma}', \quad \boldsymbol{\varepsilon} = \mathbf{T}_\varepsilon\, \boldsymbol{\varepsilon}'$$

  - From stress strain law we get

$$\mathbf{C}' = \mathbf{T}_\sigma^{-1} \mathbf{C} \mathbf{T}_\varepsilon$$

  - From strain energy function we get

$$\mathbf{C}' = \mathbf{T}_\varepsilon^{t} \mathbf{C} \mathbf{T}_\varepsilon$$

Plane stress

- For isotropic material under plane stress

$$\psi = \frac{E}{1-\nu^2}\left(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_x + \frac{1-\nu}{2}\gamma_{xy}^2\right)$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1-\nu^2}\begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}\begin{pmatrix} \varepsilon_x \\ \sigma_y \\ \gamma_{xy} \end{pmatrix}$$

- Note that ε_z is not necessarily zero