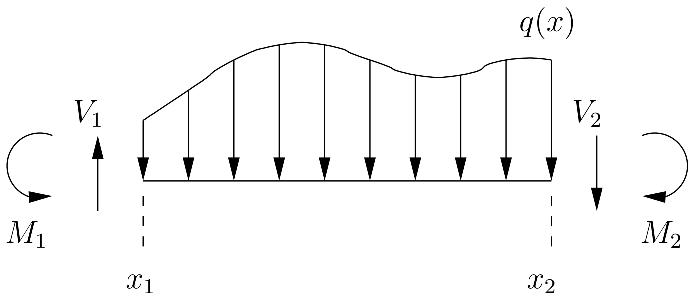
Tangents to the bending moment diagram



Consider a part of a beam, loaded by a general distributed load $q(x)$. At location $x = x_1$ we have the internal shear force V_1 and bending moment M_1. At location $x = x_2$ we have the internal shear force V_2 and the bending moment M_2. We are interested in the relation between the lines tangent to the bending moment diagram in $x = x_1$ and $x = x_2$ and the location of the resultant of $q(x)$.

Consider the line tangent to the bending moment diagram in $x = x_1$. This line is given as

$$f(x) = \left(\frac{dM}{dx}(x = x_1) \right) x + C_1 \tag{1}$$

From the differential relation $V = \dfrac{dM}{dx}$ we conclude that $\dfrac{dM}{dx}(x = x_1) = V_1$ (the derivative of M with respect to x evaluated in $x = x_1$). The tangent line is therefore given by

$$f(x) = V_1 x + C_1 \tag{2}$$

Since this line is tangent to the moment diagram in $x = x_1$, we can find the value of the constant C_1 from $f(x_1) = M_1$. This yields

$$f(x) = V_1 x + M_1 - V_1 x_1 \tag{3}$$

Similarly we have that the tangent line to the bending moment diagram in $x = x_2$ is given by

$$g(x) = \left(\frac{dM}{dx}(x = x_2) \right) x + C_2 = V_2 x + C_2 \tag{4}$$

Again from $g(x_2) = M_2$ we find $C_2 = M_2 - V_2 x_2$ which gives

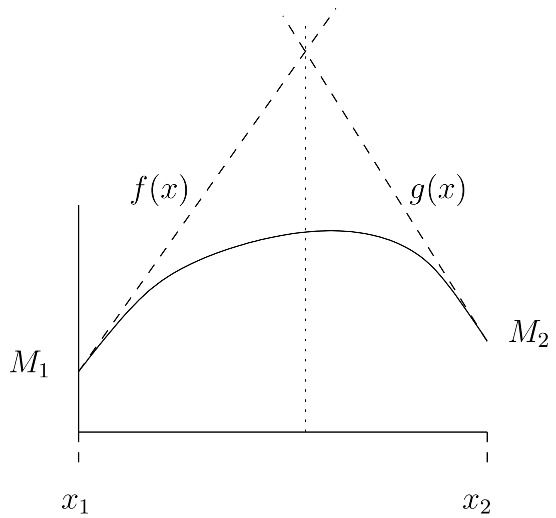
$$g(x) = V_2 x + M_2 - V_2 x_2 \tag{5}$$

The position of the intersection of both tangent lines is obtained by equating $f(x)$ and $g(x)$. This results in

$$f(x) = g(x) \Longrightarrow V_1 x + M_1 - V_1 x_1 = V_2 x + M_2 - V_2 x_2 \tag{6}$$

This then yields the position of the intersection of the tangent lines as

$$x = \frac{-M_1 + V_1 x_1 + M_2 - V_2 x_2}{V_1 - V_2} \tag{7}$$



Next, consider the equilibrium equations for the beam. From force equilibrium in the vertical direction and moment equilibrium about the left end we obtain

$$V_1 - V_2 - \int_{x_1}^{x_2} q(x)dx = 0 \tag{8}$$

and

$$-M_1 - V_2(x_2 - x_1) + M_2 - \int_{x_1}^{x_2} q(x)(x - x_1)dx = 0 \tag{9}$$

Substituting these equations into Eq. (7) yields

$$x = x_1 + \frac{\int_{x_1}^{x_2} q(x)(x - x_1)dx}{\int_{x_1}^{x_2} q(x)dx} \tag{10}$$

This is equal to the position of the resultant $R = \int_{x_1}^{x_2} q(x)dx$ of the distributed load.