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Total Potential Energy:

$\mathcal{U}$ strain energy

$\mathcal{V}$ potential energy

$$W_{in} = - \iiint \sigma_x \, \delta\varepsilon_x + \sigma_y \, \delta\varepsilon_y + \tau_{xy} \, \delta\gamma_{xy} \, dz \, dx dy$$

$$\varepsilon_x = - z \, w_{,xx} \qquad \varepsilon_y = - z \, w_{,yy} \qquad \gamma_{xy} = - 2 z \, w_{,xy}$$

$$W_{in} = + \iint \left(\underbrace{\int_{-t/2}^{t/2} \sigma_x z \, dz}_{M_x} \right) \delta w_{,xx} + \left(\underbrace{\int_{-t/2}^{t/2} \sigma_y z \, dz}_{M_y} \right) \delta w_{,yy} + 2 \left(\underbrace{\int_{-t/2}^{t/2} \tau_{xy} z \, dz}_{M_{yx}} \right) \delta w_{,xy} \, dx \, dy$$

$$W_{in} = \iint \left(M_x \, \delta w_{,xx} + M_y \, \delta w_{,yy} + 2 \, M_{yx} \, \delta w_{,xy} \right) dx dy$$

$$M_x = -D \left[w_{,xx} + \nu \, w_{,yy} \right], \quad M_y = -D \left[w_{,yy} + \nu \, w_{,xx} \right]$$

$$M_{xy} = -D (1 - \nu) \, w_{,xy}$$

$$W_{in} = - \iint D \Big[w_{,xx} \, \delta w_{,xx} + w_{,yy} \, \delta w_{,yy} + \nu \left(w_{,yy} \, \delta w_{,xx} + w_{,xx} \, \delta w_{,yy} \right) + 2 (1 - \nu) \, w_{,xy} \, \delta w_{,xy} \Big] \, dx \, dy$$

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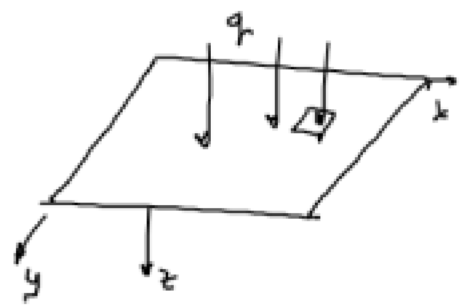
$$\delta U = -W_{in}$$

$$\delta U = \iint D\left[\underline{w_{,xx}\,\delta w_{,xx}} + w_{,yy}\,\delta w_{,yy} + \nu\left(w_{,yy}\,\delta w_{,xx} + w_{,xx}\,\delta w_{,yy}\right) + 2(1-\nu)\,w_{,xy}\,\delta w_{,xy}\right] dx\,dy$$

$$\frac{1}{2}\delta w_{,xx}^2 + \frac{1}{2}\delta w_{,yy}^2 + \nu\,\delta\left(w_{,xx}\,w_{,yy}\right) + (1-\nu)\,\delta w_{,xy}^2$$

$$\delta U = \delta \underbrace{\iint \frac{D}{2}\left[w_{,xx}^2 + w_{,yy}^2 + 2\nu\,w_{,xx}\,w_{,yy} + 2(1-\nu)\,w_{,xy}^2\right] dx\,dy}_{\text{strain energy due to bending}}$$

$$W_e = \iint q\,\delta w\,dx\,dy$$



$$\delta V = -\iint q\,\delta w\,dx\,dy$$

$$\underbrace{V = -\iint q\,w\,dx\,dy}_{\text{potential energy.}}$$

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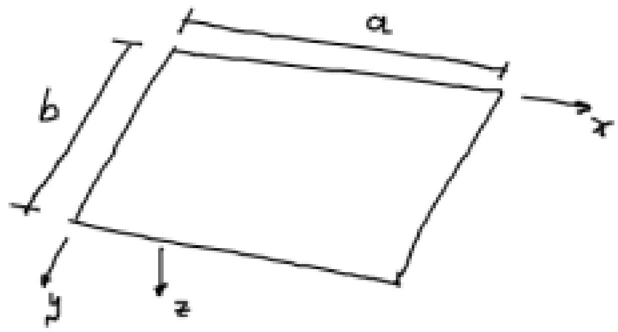
Eqm equations

$$\delta(U+V) = 0$$

$$0 = -\iint M_x\,\delta w_{,xx} + M_y\,\delta w_{,yy} + 2M_{yx}\,\delta w_{,xy}\,dx\,dy - \iint q\,\delta w\,dx\,dy$$

$$\iint \left[M_x\,\delta w_{,xx} + M_y\,\delta w_{,yy} + 2M_{yx}\,\delta w_{,xy} + q\,\delta w\right] dx\,dy = 0$$

$$D\left[w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}\right] = q$$



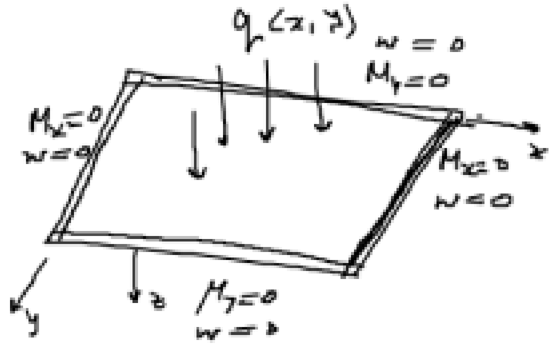
edges $\perp$ x

1- $w_{,x} = 0$ OR $M_x = 0$

2- $w = 0$ OR $V_x = 0$

$$V_x = Q_x + \frac{\partial M_{yx}}{\partial y}$$

Navier Solution



$$D\left[w_{,xxxx} + 2 w_{,xxyy} + w_{,yyyy} \right] = q$$

$\perp x \,|\quad w = 0 \qquad w_{,xx} = 0$

$\perp y \,|\quad w = 0 \qquad w_{,yy} = 0$

$$w = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$w = C_1 \sin\alpha x + C_2 \cos\beta x$$

$w(0) = 0, \qquad C_2 = 0$

$w(a) = 0, \qquad C_1 \sin\alpha a = 0$

$$\alpha a = m\pi$$

$$w = C_1 \sin\frac{m\pi x}{a}$$

$w(a,y) = 0 \qquad w_{yy}(a,y) = 0$

$M_x(a,y) = 0$

$-D\left[w_{,xx} + \nu w_{,yy} \right] = 0$

$w_{,xx}(a,y) = 0$

$$D\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4 \right] A_{mn} \underbrace{\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}} = q$$

if $\big|\; q = a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$

$$A_{mn} = \frac{a_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \right]^2} \longrightarrow \#$$

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$$q(x,y) = \sum_{m,n} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$a_{mn} = \frac{4}{ab}\int_0^b\int_0^a q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \rightarrow *$$

$$w = \sum_{m,n} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

1- find the sine coeff. of q

2- Calculate the sine coeff of w

3- find w

4- find M_x, M_y, M_{yx} by differentiation

5- find σ_x, σ_y, τ_{xy}

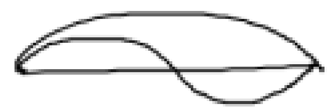
1- uniform pressure q_o

$$a_{mn} = \frac{4}{ab} q_o \int_0^b \left(\int_0^a \sin \frac{m\pi x}{a} \, dx \right) \sin \frac{n\pi y}{b} \, dy$$

$$a_{mn} = \frac{4 q_o}{ab} \left(\int_0^a \sin \frac{m\pi x}{a} \, dx \right) \left(\int_0^b \sin \frac{n\pi y}{b} \, dy \right)$$

$$\int_0^a \sin \frac{m\pi x}{a} \, dx = -\frac{a}{m\pi} \cos \frac{m\pi x}{a} \Big|_0^a = -\frac{a}{m\pi} \cos m\pi + \frac{a}{m\pi}$$

$$= \frac{a}{m\pi} \left[1 - \cos m\pi \right] = \begin{cases} 0 & m \text{ even} \\ \frac{2a}{m\pi} & m \text{ odd} \end{cases}$$



$$a_{mn} = \begin{cases} \frac{16 q_o}{\pi^2 mn} & m \,\&\, n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

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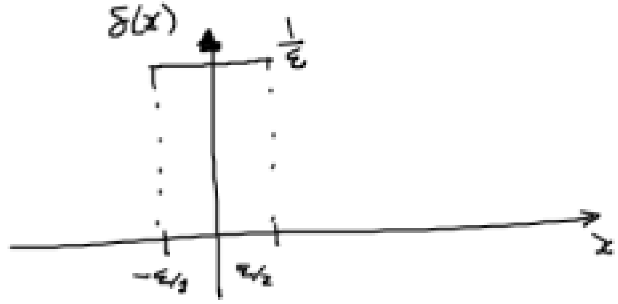
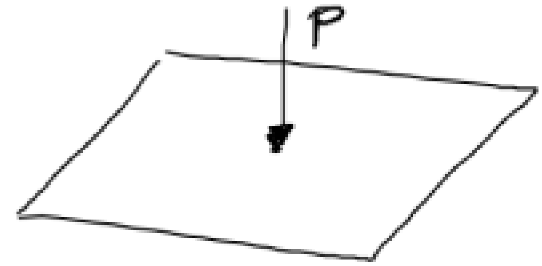
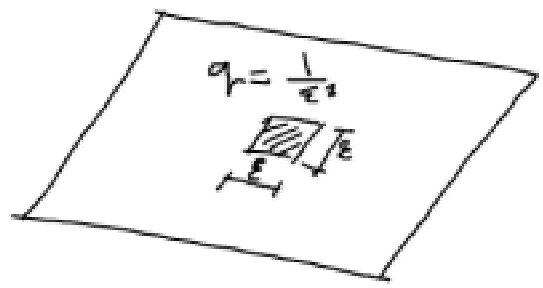
2- Concentrated force





$$\int f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

$$q(x,y) = P\begin{cases} \frac{1}{\varepsilon^2} & x_0-\frac{\varepsilon}{2} \le x \le x_0+\frac{\varepsilon}{2},\ y_0-\frac{\varepsilon}{2} \le y \le y_0+\frac{\varepsilon}{2} \\ 0 & \text{else} \end{cases}$$



$$q(x,y) = P\underbrace{\left(\begin{cases} \frac{1}{\varepsilon} & x_0-\frac{\varepsilon}{2} \le x \le x_0+\frac{\varepsilon}{2} \\ 0 & \text{else} \end{cases}\right)}_{\delta(x-x_0)}\underbrace{\left(\begin{cases} \frac{1}{\varepsilon} & y_0-\frac{\varepsilon}{2} \le y \le y_0+\frac{\varepsilon}{2} \\ 0 & \text{else} \end{cases}\right)}_{\delta(y-y_0)}$$

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$$q = P\,\delta(x - x_0)\,\delta(y - y_0)$$

$$a_{mn} = \frac{4}{ab}\int_0^b\int_0^a P\,\delta(x - x_0)\,\delta(y - y_0)\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,dx\,dy$$

$$a_{mn} = \frac{4P}{ab}\left(\int_0^a \sin\frac{m\pi x}{a}\,\delta(x - x_0)\,dx\right)\left(\int_0^b \sin\frac{n\pi y}{b}\,\delta(y - y_0)\,dy\right)$$

$$a_{mn} = \frac{4P}{ab}\,\sin\frac{m\pi x_0}{a}\,\sin\frac{n\pi y_0}{b}$$

$$A_{mn} = \frac{a_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2}$$