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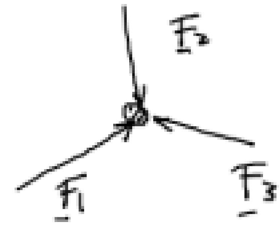
Principle of virtual work:

For a particle



$$\underline{F} = \sum \underline{F}_i = \underline{0}$$

$$\underline{F} \cdot \delta\underline{u} = 0 \longrightarrow \#$$

$$\delta\underline{u} \propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} F_x &= 0 \\ F_y &= 0 \\ F_z &= 0 \end{aligned}\right\} \qquad \underline{F} = 0$$

A particle is in eq^m^ under the action of forces <u>if and only if</u> the work done by the forces on any arbitrary <u>virtual</u> displacement is zero

$$\underline{F}^t \delta\underline{u} = 0 \qquad \delta\underline{u}^t \underline{F} = 0$$

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PVW for a system of particles

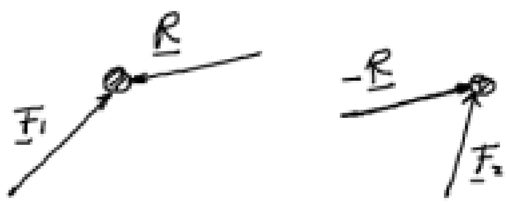
$$W = \underline{F_1}^t \delta \underline{u_1} + \underline{F_2}^t \delta \underline{u_2}$$

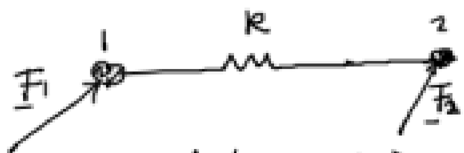
$$W = 0$$

$$\underline{F_1} = \underline{0} \qquad \underline{F_2} = \underline{0}$$

$$W = (\underline{F_1} + \underline{R})^t \delta \underline{u_1} + (\underline{F_2} - \underline{R})^t \delta \underline{u_2}$$

$$W = \underbrace{\underline{F_1}^t \delta \underline{u_1} + \underline{F_2}^t \delta \underline{u_2}}_{W_{ex}} + \underbrace{\underline{R}^t (\delta \underline{u_1} - \delta \underline{u_2})}_{W_{in}}$$





$$R_x = k(u_{1x} - u_{2x})$$
$$R_y = 0$$

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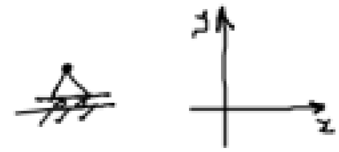
$\underline{R}^t(\delta\underline{u}_1 - \delta\underline{u}_2)$!?

$= 0$



$\delta\underline{u} = \begin{pmatrix} c \\ 0 \end{pmatrix}$

$\underline{R} = \begin{pmatrix} 0 \\ R \end{pmatrix}$

$\underline{R}^t \delta\underline{u} = 0$

<u>for rigid connection</u>

$\underline{R}$: aligned with the line connecting the two particles

$$\left.\begin{array}{l}\delta\underline{u}_1 - \delta\underline{u}_2 : \text{has zero component} \\ \qquad\text{along the same line}\end{array}\right\} \text{geometric condition}$$

$\underline{R} = \begin{pmatrix} R \\ 0 \end{pmatrix}$ $\quad \delta\underline{u}_1 - \delta\underline{u}_2 = \begin{pmatrix} 0 \\ c \end{pmatrix}$

$$W = W_{ex} + \underbrace{\cancel{W_{in}}}_{0} = \underline{F}_1^t \delta\underline{u}_1 + \underline{F}_2^t \delta\underline{u}_2 = 0$$

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PVW:-

$$W = W_{ex} + W_{in}$$

$\delta \underline{u}$:

1- arbitrary
2- small (infinitesimally small)
3. Compatible with geometric Constraints

for rigid bodies $W_{in} = 0$

for deformable bodies $W_{in} \neq 0$

$$\boxed{W = 0}$$

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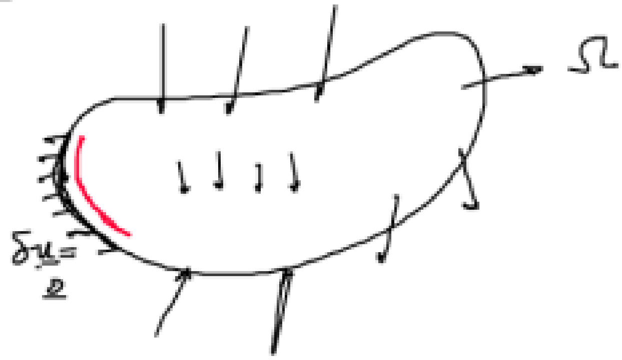
PVW for deformable bodies

$\underline{b}$: body force per unit volume

$\underline{t}$: force per unit area



$$W_{ex} = \int_{\Omega} \underline{b}^t \, \delta \underline{u} \, dV + \int_{\partial\Omega} \underline{t}^t \, \delta \underline{u} \, dA$$

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$$W_t = \int_{\partial\Omega} \underline{t}^t \, \delta\underline{u} \; dA$$

$$\underline{t} = \underline{\underline{\sigma}} \; \underline{n}$$

$$\underline{t}^t \, \delta\underline{u} = \underline{n}^t \, \underline{\underline{\sigma}} \, \delta\underline{u} = \left(\delta\underline{u}^t \, \underline{\underline{\sigma}}\right)\underline{n}$$

$$W_t = \int_{\partial\Omega} \left(\underline{\underline{\sigma}} \, \delta\underline{u}\right)^t \underline{n} \; dA$$

$$= \int_{\Omega} \underline{\nabla} \cdot \left(\underline{\underline{\sigma}} \, \delta\underline{u}\right) \; dV$$

$$\underline{\nabla} \cdot \left(\underline{\underline{\sigma}} \, \delta\underline{u}\right) = \left(\underline{\nabla} \cdot \underline{\underline{\sigma}}\right)^t \delta\underline{u} + \underline{\underline{\sigma}} : \underline{\nabla} \, \delta\underline{u}$$

$$W_t = \int_{\Omega} \left(\underline{\nabla} \cdot \underline{\underline{\sigma}}\right)^t \delta\underline{u} \, dV + \int_{\Omega} \underline{\underline{\sigma}} : \underline{\nabla} \, \delta\underline{u} \; dV$$

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$$W_{ex} = \int_\Omega \underline{b}^t \delta \underline{u} \, dV + \int_\Omega (\underline{\nabla} \cdot \underline{\underline{\sigma}})^t \delta \underline{u} \, dV + \underbrace{\int_\Omega \underline{\underline{\sigma}} : \underline{\nabla} \delta \underline{u} \, dV}$$

$$W_{ex} - \int_\Omega \underline{\underline{\sigma}} : \underline{\nabla} \delta \underline{u} \, dV = \int_\Omega (\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{b})^t \delta \underline{u} \, d\Omega$$

$$= 0$$

$$W_{in} = - \int_\Omega \underline{\underline{\sigma}} : \underbrace{(\underline{\nabla} \delta \underline{u})}_{\delta \underline{\underline{H}}} \, dV$$

$$\delta \underline{\underline{H}} = \delta \underline{\underline{\varepsilon}} + \delta \underline{\underline{\phi}}$$

$$\boxed{W_{in} = - \int_\Omega \underline{\underline{\sigma}} : \delta \underline{\underline{\varepsilon}} \, dV}$$

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Work done by internal forces

$$\delta\underline{\underline{\varepsilon}} = \begin{pmatrix} \delta\varepsilon_x & \delta\gamma_{xy}/2 \\ \frac{\delta\gamma_{xy}}{2} & \delta\varepsilon_y \end{pmatrix} \qquad \underline{\underline{\sigma}} = \begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix}$$

$$\underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}} = \sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \tau_{xy}\frac{\delta\gamma_{xy}}{2} + \tau_{xy}\frac{\delta\gamma_{xy}}{2}$$

$$\underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}} = \sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \tau_{xy}\,\delta\gamma_{xy}$$

$$\underline{\underline{\sigma}} : \delta\underline{\underline{\varepsilon}} = \sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{yz}\,\delta\gamma_{yz} + \tau_{xz}\,\delta\gamma_{xz} + \tau_{xy}\,\delta\gamma_{xy}$$

$$\underline{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{pmatrix} \qquad \underline{\varepsilon} = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{pmatrix} \qquad W_{in} = -\int_\Omega \underline{\sigma}^t\,\delta\underline{\varepsilon}\,dV$$