Dynamics and Stability AE3-914

Answers to recommended exercises—Week 4

6.1 The equilibrium points and their nature are:
(a) $x = 0$ is unstable; $x = \pm 1$ are stable
(b) $x = 0$ is stable
(c) $x = 0$ is unstable; $x = 1$ is stable

Hint: The equilibrium points are found from $\dot{x} = \dot{y} = 0$, where $y = \dot{x}$. The characteristics of each equilibrium point are found from the eigenvalues of the matrix corresponding to the linearisation of the system of differential equations provided by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= f(x, y) \end{aligned}$$

where f is problem dependent. Alternatively, the potential V can be analysed for maxima and minima on the equilibrium points.

6.14 If $\omega^2 < k/m$ the system is stable and if $\omega^2 > k/m$ it is unstable. If a damper with coefficient c is added, the unstable configuration remains unstable because there is always a positive eigenvalue. The stable configuration can exhibit two different cases:

If $0 < (c/m)^2 + 4(\omega^2 - k/m) < (c/m)^2$ the eigenvalues are real and negative and the system is overdamped. Qualitatively, there will be no oscillation about the equilibrium position.

If $(c/m)^2 + 4(\omega^2 - k/m) < 0$ the eigenvalues have a negative real part and an imaginary part, which situation corresponds to a subcritically damped system. There will be an oscillation about the equilibrium position.

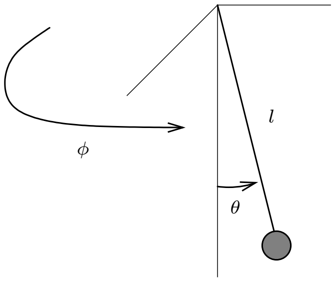
Hint: The first case can be solved on an effective potential, but the second case must be solved by linearisation of the equations of motion (why?).

6.16 The equilibrium position $\theta = 0$ is stable if $P < 2K/L$, where $L = OA = AB$ and θ is the angle between OA and OB.

Hint: When stating the equation of motion, notice that P is an applied force that should be converted to a generalised force conjugate to the coordinate θ by means of, e.g., the method of virtual work. Notice also that the angle entering the elastic potential energy is 2θ rather than θ. Use a method of your choice, but use it *properly!*

6.29 Steady motion is found when

$$\cos\theta = \frac{g}{l\dot{\phi}^2}$$



The eigenvalues of the linearised equation of motion in θ are always negative, so the position is always stable. Notice that the position $\theta = 0$ is attained when $\dot{\phi}^2 = g/l$. For initial values of $\dot{\phi}^2$ smaller than g/l the angle θ would be undefined. Physically this would correspond to a position in which the pendulum remains vertical but which cannot be viewed as steady motion, because in such a case the integral of motion corresponding to the coordinate ϕ would identically vanish and the equation for θ would degenerate to that of a simple pendulum.

Warning: A possible error could be to assume that $\dot{\phi}$ is constant. In such a case we get three values of θ for which an equilibrium situation is possible:

$$\theta = 0, \quad \cos\theta = \frac{g}{l\dot{\phi}^2} \quad \text{and} \quad \theta = \pi$$

The position $\theta = \pi$ is always unstable. The position $\theta = 0$ is stable when $\dot{\phi}^2 < g/l$ and unstable when $\dot{\phi}^2 > g/l$. The position $\cos\theta = g/(l\dot{\phi}^2)$ is always stable. Notice that this position is only possible when the position $\theta = 0$ has become unstable. The difference with the actual case is that if $\dot{\phi}$ is enforced to be constant we cannot have conservation of angular momentum when θ changes and that, formally, we cannot speak of steady motion.

6.31 The position $\theta = 0$ is stable when $mgl < 2ka^2$, where l is the length of the pendulum.

6.43 The vertical solution is stable when

$$mg > \frac{k(2a - l)}{4}$$

6.46 The bar is horizontal when $\theta = 0$ or $\theta = \pi$. Notice that the bar rotates with constant angular velocity ω about a vertical, which makes this problem essentially different from 6.29. A careful analysis of the eigenvalues of the linearised equation around the equilibrium points (or of the maxima/minima of the effective potential) reveals the following:

For $\theta = 0$:

If $a > l/\sqrt{3}$ then the position is stable when

$$\omega^2 < \frac{3g\sqrt{a^2 - \frac{1}{4}l^2}}{3a^2 - l^2}$$

If $l/2 < a < l/\sqrt{3}$ then the position is stable when

$$\omega^2 > \frac{3g\sqrt{a^2 - \frac{1}{4}l^2}}{3a^2 - l^2}$$

Since the right hand term in the latter equation is always negative in the corresponding interval of a, it can be stated that it will always be smaller than ω^2. So, in such a case, the position is stable for any ω.

For $\theta = \pi$:

If $a > l/\sqrt{3}$ the position would be stable when

$$\omega^2 < \frac{-3g\sqrt{a^2 - \frac{1}{4}l^2}}{3a^2 - l^2}$$

but since the right hand side is negative in such a case, no ω can satisfy this equation. So the position is always unstable.

If $l/2 < a < l/\sqrt{3}$ the position is stable when

$$\omega^2 > \frac{-3g\sqrt{a^2 - \frac{1}{4}l^2}}{3a^2 - l^2}$$

where the right hand side is positive. Thus, for sufficiently large ω the upper horizontal position is stable.

It is left to the reader to provide an intuitive interpretation of why the predicted stable configurations are actually stable.