Dynamics and Stability AE3-914

Answers to recommended exercises—Week 6

3.3

(a) $\delta F = 10x\delta y + 2xy'\delta y'$

(b) $\delta I = \int_{x_1}^{x_2} [10x - 2(y' + xy'')]\delta y \, dx + 2xy'\delta y\Big|_{x_1}^{x_2}$

(c) $10x - 2(y' + xy'') = 0$

3.5

(a) $2y'' + 1 = 0$

(b) $y'' - y = e^x$

(c) $y = -x/2$

(d) $y = x$

(e) $y'' + 16y = 0$

(f) $x^2y'' + 2xy' - 2y = x$

3.9 Euler-Lagrange equation:

$$E\mathcal{I}\frac{d^4y}{dx^4} - q = 0$$

Natural boundary conditions:

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = \left.\frac{d^2y}{dx^2}\right|_{x=L} = 0$$

If $E = E(x)$ and $\mathcal{I} = \mathcal{I}(x)$ the expressions in the variation of the functional will become more complicated. The Euler-Lagrange equations become

$$E\mathcal{I}\frac{d^4y}{dx^4} + 2\frac{d(E\mathcal{I})}{dx}\frac{d^3y}{dx^3} + \frac{d^2(E\mathcal{I})}{dx^2}\frac{d^2y}{dx^2} - q = 0$$

but the natural boundary conditions do not change.

Hint: Use the standard integration by parts procedure and find the natural boundary conditions from that.

3.32 That is essentially the same as finding the shortest path between two points on a plane, so it should not be difficult for you. Notice that the book forgot to mention that the helix might include an integration constant.

3.34(a)

Euler-Lagrange equation:

$$y'' = y$$

Essential boundary condition:

$$y(0) = 1$$

Natural boundary condition:

$$y'(1) = 0$$

Solution:

$$y = \frac{\cosh(1-x)}{\cosh 1}$$

3.34(b)

Euler-Lagrange equation:

$$y'' = 1$$

Essential boundary condition:

$$y(0) = \frac{1}{2}$$

Natural boundary condition:

$$y'(1) + y(1) = -1$$

Solution:

$$y = \frac{1}{2}(x^2 - 3x + 1)$$

Hint: Do not rely on given formulas for natural boundary conditions but find them on the variation of the functionals.

Remark: Notice that the concepts "essential" and "natural" are not coupled to the kind of boundary condition of the differential equation (Dirichlet, Neumann or Robin).

Extra exercise:

Show that if a functional

$$I(y) = \int F\,dx$$

is such that the integrand does not depend on x, that is, $F = F(y, y')$ then the extremals of I are the solution of the first-order differential equation

$$F - y'\frac{\partial F}{\partial y'} = C,$$

where C is a constant.

See next page for hints (after thinking by yourself for a while!)

Hint: The proposed integrand is analogous to a Lagrangian not depending on time. There is something about the Jacobi integral and integrals of motion for such a Lagrangian. The derivation asked for would be analogous to that in page 123 of Török. Rewrite it for this case.