Elaboration Exam "Dynamics and Stability" (AE3-914) of August 23, 2010.

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Question 1:

a.) $\bar{v} = \bar{v}_{xyz} + \bar{\omega}_{xyz} \times \bar{r}_{REL} + \bar{v}_{REL}$

$\bar{r}_{REL} = l\cos\theta\,\bar{i} + l\sin\theta\,\bar{j}$

$\bar{v}_{REL} = \dot{\bar{r}}_{REL} = -l\sin\theta\,\dot{\theta}\,\bar{i} + l\cos\theta\,\dot{\theta}\,\bar{j}$

$\bar{\omega}_{xyz} = \dot{B}\bar{J} = \dot{B}\cdot\bar{j}$

$\bar{v}_{xyz} = \dot{B}\cdot C\,(\sin B\,\bar{I} + \cos B\,\bar{K})$

The axes of the inertial frame of reference X-Y-Z and the non-inertial frame of reference x-y-z are related as:

$X = \cos B\,x + \sin B\,z$
$Z = \cos B\,z - \sin B\,x$

Hence, for the corresponding unit vectors similar relations apply, so that $\bar{v}_{xyz}$ may be written as:

$$\bar{v}_{xyz} = \dot{B}C\left[(\cos B\,\bar{i} + \sin B\,\bar{k}) * \sin B + \cos B * (\cos B\,\bar{k} - \sin B\,\bar{i})\right]$$

$$= \dot{B}C\bar{k}$$

Combining the above expressions gives:

$$\bar{v} = \mathring{B} c \bar{k} + \mathring{B} \bar{j} \times (l\cos\theta\, \bar{i} + l\sin\theta\, \bar{j})$$
$$- l\sin\theta\, \mathring{\theta}\, \bar{i} + l\cos\theta\, \mathring{\theta}\, \bar{j}$$

$$\Rightarrow \bar{v} = \mathring{B} c \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & \mathring{B} & 0 \\ l\cos\theta & l\sin\theta & 0 \end{vmatrix} - l\sin\theta\, \mathring{\theta}\, \bar{i}$$
$$+ l\cos\theta\, \mathring{\theta}\, \bar{j}$$

$$\Rightarrow \bar{v} = \mathring{B} c \bar{k} - \mathring{B} l\cos\theta\, \bar{k} - l\sin\theta\, \mathring{\theta}\, \bar{i} + l\cos\theta\, \mathring{\theta}\, \bar{j}$$

$$\Rightarrow \boxed{\bar{v} = l\mathring{\theta}(-\sin\theta\, \bar{i} + \cos\theta\, \bar{j}) + \mathring{B}(c - l\cos\theta)\, \bar{k}}$$

b.) $T = \frac{1}{2} \cdot m\, (\bar{v} \cdot \bar{v})$

$$\Rightarrow \boxed{T = \frac{1}{2} m \left(l^2 \mathring{\theta}^2 + \mathring{B}^2 (c - l\cos\theta)^2 \right)}$$

using the fact that $\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$
and $\bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{i} = 0$

c.) $L = T - V$;

$$V = m \cdot g \cdot (c - l\cos\theta) \cdot \sin B$$

$$\Rightarrow \boxed{\begin{aligned} L = {} & \frac{1}{2} m \left(l^2 \mathring{\theta}^2 + \mathring{B}^2 (c - l\cos\theta)^2 \right) \\ & - mg (c - l\cos\theta) \sin B \end{aligned}}$$

d.) The equations of motion can be constructed from the Lagrangian L, thereby also accounting for possible constraint equations.

The constraint equation for the problem follows from the fact that the mass m and the inclined plane remain in contact during motion. Essentially, this comes down to prescribing the velocity of the mass to be equal to the velocity of the inclined plane, in the direction ⊥ to the inclined plane (the $\bar{k}$-direction) at their point of contact. Accordingly, from the expression for $\bar{v}$ found under a.) this constraint follows as:

$$V_z = \bar{V} \cdot \bar{k} = \dot{B}(c - l\cos\theta) .$$

Note that in the $\bar{i}$ and $\bar{j}$ directions there are <u>no</u> constraints, since the inclined plane is <u>frictionless</u>.

The above constraint equation can be rewritten as

$$\boxed{(c - l\cos\theta)\mathring{B} - V_z = 0}$$

Comparing this relation with the formal representation (Eq. 2.71 on page 119 of Török):

$$A_1 \mathring{q}_1 + A_2 \mathring{q}_2 + \dots + A_n \mathring{q}_n + A_0 = 0,$$

where the generalized coordinates are interpreted as

$$q_1 = B \quad \text{and} \quad q_2 = \theta,$$

gives for the non-holonomic constraint:

$$A_1 \mathring{q}_1 + A_0 = 0$$

where

$$A_1 = A_B = c - l\cos\theta$$
$$\mathring{q}_1 = \mathring{B}$$
$$A_0 = -V_z.$$

Using the Lagrangian L and the above constraint, 2 equations of motion can be formulated with respect to the generalised coordinates θ an B, respectively.

For the generalised coordinate θ we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Note that the constraint equation, which relates to $\delta \beta$, does not affect this equation of motion. From the expression of L under c.), we obtain

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \quad ; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta} ;$$

$$\frac{\partial L}{\partial \theta} = m \dot{\beta}^2 (c - l\cos\theta) \cdot l \sin\theta - m g l \sin\theta \sin\beta .$$

Hence, the equation of motion becomes:

$$m l^2 \ddot{\theta} - m \dot{\beta}^2 (c - l\cos\theta) l \sin\theta + m g l \sin\theta \sin\beta = 0$$

which can be simplified as

$$\boxed{l\ddot{\theta} - \dot{\beta}^2 (c - l\cos\theta)\sin\theta + g\sin\theta\sin\beta = 0}$$

Next, for the generalised coordinate β we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\beta}}\right) - \frac{\partial L}{\partial \beta} = \lambda_\beta A_\beta$$

where the right-hand side reflects the contribution related to the constraint equation, with λ_B a Lagrange multiplier.

Since,

$$\frac{\partial L}{\partial \dot{B}} = m \dot{B} (c - l\cos\theta)^2 ;$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{B}}\right) = m \ddot{B} (c - l\cos\theta)^2 + 2m\dot{B}(c - l\cos\theta) \cdot l\sin\theta \, \dot{\theta} ;$$

$$\frac{\partial L}{\partial B} = -mg (c - l\cos\theta) \cos B ,$$

and $A_B = c - l\cos\theta$, the equation of motion becomes

$$m\ddot{B}(c - l\cos\theta)^2 + 2m\dot{B}(c - l\cos\theta) \, l\sin\theta \, \dot{\theta} + mg (c - l\cos\theta)\cos B = \lambda_B (c - l\cos\theta)$$

Dividing this expression by $(c - l\cos\theta)$ leads to:

$$\boxed{m\ddot{B}(c - l\cos\theta) + 2m\dot{B} \, l\sin\theta \, \dot{\theta} + mg\cos B = \lambda_B}$$

The Lagrange multiplier λ_B thus represents the contact force between the mass m and the inclined plane, in the direction $\perp$ to the plane of contact (i.e., the normal contact force).

In summary, the system behaviour is described by 2 generalised coordinates $\theta(t)$ and $B(t)$ and 1 constraint equation related to the preservation of contact between the mass and the inclined plane.

The number of degree of freedom thus is:

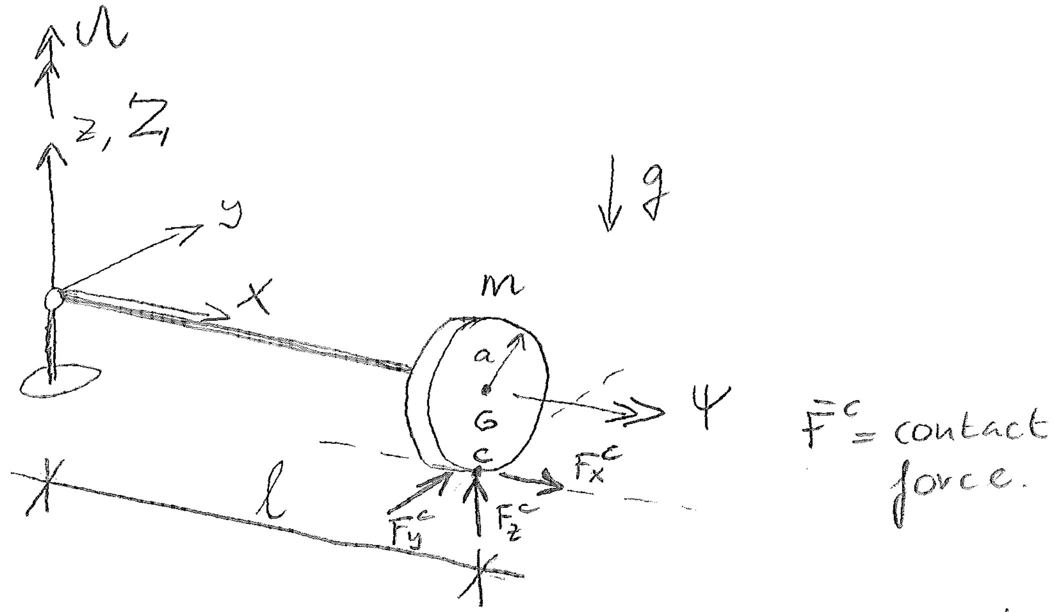
$$\# \text{ generalised coordinates} - \# \text{ constraints}$$

$$= 2 - 1 = 1 \quad (\text{i.e. } \theta).$$

Note that the 2 equations of motion above need to be solved simultaneously (due to their coupling) for the 2 unknowns $\ddot{\theta}$ and the normal contact force λ_B (i.e., the rotation $B(t)$ is prescribed and therefore not an unknown!).

Question (2) :

a)



Note that the x-axis represents a symmetry axis for the wheel. Accordingly, the mass moments of inertia w.r.t. the centre of mass G can be formulated as:

$$I_{xx}^G = I_s^G = \frac{1}{2} m a^2$$

$$I_{yy}^G = I_{zz}^G = I^G = \frac{1}{4} m a^2$$

To formulate the angular momentum, the auxilirary rotational velocity ψ about the x-as has been introduced, see above figure.

Since the wheel rolls <u>without</u> slipping, Ω and ψ are the rotational velocities related as:

$$\Omega \cdot l = -\psi \cdot a \qquad (= V_y)$$

from which ψ is obtained as

$$\boxed{\psi = \frac{-l}{a} \cdot \Omega}$$

The angular momentum about point G can now be written as

$$\bar{L}_G = I_s^G \, \psi \, \bar{i} + I^G \, \Omega \, \bar{k}$$

$$= \frac{1}{2} m a^2 \cdot \frac{-l}{a} \cdot \Omega \, \bar{i} + \frac{1}{4} m a^2 \, \Omega \, \bar{k}$$

$$= \frac{1}{4} m a^2 \, \Omega \left(\frac{-2l}{a} \, \bar{i} + \bar{k} \right)$$

Now, the angular momentum about point O can be found using the expression (Page 206, book of Török):

$$\bar{L}_O = \bar{r}_{OG} \times \bar{p} + \bar{L}_G$$

Here $\bar{p}$ is the linear momentum of the wheel, given by

$$\bar{p} = m \, (\Omega l) \, \bar{j}$$

Furthermore, the position vector of the mass centre of the wheel is:

$$\bar{r}_{OG} = l \, \bar{i} .$$

Hence,

$$\bar{L}_O = l \bar{i} \times m l \Omega \bar{j} + \frac{1}{4} m a^2 \Omega \left(-\frac{l}{a} \bar{i} + \bar{k} \right)$$

$$\Rightarrow \boxed{\bar{L}_O = \Omega \left(-\frac{1}{2} m a l \, \bar{i} + \left(\frac{1}{4} m a^2 + m l^2 \right) \bar{k} \right)}$$

b.) The equation of motion about point O can be written as:

$$\dot{\bar{L}}_o = \Sigma \bar{M}_o$$

where $\bar{L}_o$ is the answer of a.).

Accordingly, $\dot{\bar{L}}_o$ is obtained as:

$$\dot{\bar{L}}_o = \dot{\Omega}\left(-\tfrac{1}{2}mal\,\bar{i} + \left(\tfrac{1}{4}ma^2 + ml^2\right)\bar{k}\right) + \bar{\omega} \times \left(\Omega\left(-\tfrac{1}{2}mal\,\bar{i} + \left(\tfrac{1}{4}ma^2 + ml^2\right)\bar{k}\right)\right)$$

where $\bar{\omega}$ represents the rotation of the moving frame of reference x-y-z, i.e.,

$$\bar{\omega} = \Omega \bar{K} = \Omega \bar{k}$$

Substituting the rotation in the expression above leads to

$$\boxed{\dot{\bar{L}}_o = \dot{\Omega}\left(-\tfrac{1}{2}mal\,\bar{i} + \left(\tfrac{1}{4}ma^2 + ml^2\right)\bar{k}\right) - \Omega^2 \tfrac{1}{2}mal\,\bar{j}}$$

(note that: $\bar{k} \times \bar{i} = \bar{j}$ and $\bar{k} \times \bar{k} = \bar{0}$)

However, since Ω = constant, and thus $\dot{\Omega} = 0$, the above expression reduces to

$$\boxed{\dot{\bar{L}}_o = -\Omega^2 \tfrac{1}{2}mal\,\bar{j}}$$

Next, (11)

$$\Sigma \bar{M}_o = \bar{r}_{oc} \times \bar{F}_c + \bar{r}_{oG} \times \bar{W}$$

where $\bar{F}_c$ is the contact force at the contact point C between the wheel and the horizontal plane, i.e.,

$$\bar{F}_c = F_x^c \, \bar{i} + F_y^c \, \bar{j} + F_z^c \, \bar{k},$$

and $\bar{W}$ is the gravitational loading at the centre of mass G,

$$\bar{W} = -m \cdot g \cdot \bar{k}.$$

With the position vectors given by

$$\bar{r}_{oc} = l\bar{i} - a\bar{k},$$

$$\bar{r}_{oG} = l\bar{i},$$

we obtain

$$\Sigma \bar{M}_o = (l\bar{i} - a\bar{k}) \times (F_x^c \bar{i} + F_y^c \bar{j} + F_z^c \bar{k}) + l\bar{i} \times (-mg\bar{k})$$

$$\Rightarrow \Sigma \bar{M}_o = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ l & 0 & -a \\ F_x^c & F_y^c & F_z^c \end{vmatrix} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ l & 0 & 0 \\ 0 & 0 & -mg \end{vmatrix}$$

$$\Rightarrow \boxed{\sum \bar{M}_0 = F_y^c\, a\, \bar{i} + (-a F_x^c - l F_z^c + mgl)\bar{j} + F_y^c\, l\, \bar{k}}$$

Now, from the basic relation $\dot{\bar{L}}_0 = \sum \bar{M}_0$, and using the expressions obtained for $\dot{\bar{L}}_0$ and $\sum \bar{M}_0$, we finally obtain that

$$F_y^c = 0 \; ; \qquad \boxed{\frac{1}{2} m a l \Omega^2 = a F_x^c + (F_z^c - mg) l}$$

Question (3):

The energy functional of the system is

$$V = \int_0^L \left(-m\phi + \frac{1}{2} G I_p \phi_x^2\right) dx - M\phi(L)$$

This expression follows from the fact that the virtual work generated by the external loads (moments) is

$$\delta W = \int_0^L m\, \delta\phi\, dx + M\,\delta\phi(L)$$

For conservative systems we may write

$$\delta W = -\delta V^{\text{loading}}$$

with $V^{loading}$ the potential energy caused by the loading, i.e.,

$$V^{loading} = \int_0^L -m\phi \, dx - M\phi(L).$$

This potential energy may be added to the strain energy

$$V^{strain} = \int_0^L \tfrac{1}{2} G I_p \phi_x^2 \, dx$$

to obtain the total potential energy V:

$$V = V^{loading} + V^{strain}$$

$$\Rightarrow \quad V = \int_0^L \left(-m\phi + \tfrac{1}{2} G I_p \phi_x^2\right) dx - M\phi(L).$$

a.) For a static problem, equilibrium is defined by the potential energy being stationary, i.e.,

$$\delta V = 0.$$

With the above energy functional this leads to

$$\delta V = \int_0^L \left(-m\,\delta\phi + G I_p \phi_x \,\delta\phi_x\right) dx - M\,\delta\phi(L) = 0$$

Using integration by parts, this expression is further developed as:

$$\delta V = GI_p \, \phi_x \, \delta\phi \Big|_0^L - \int_0^L GI_p \, \phi_{xx} \, \delta\phi \, dx - \int_0^L m \, \delta\phi \, dx - M \, \delta\phi(L) = 0$$

$$\boxed{\begin{aligned} \delta V = \; & \big(GI_p \, \phi_x(L) - M\big) \, \delta\phi(L) \quad (1) \\ & - GI_p \, \phi_x(0) \cdot \delta\phi(0) \quad (2) \\ & - \int_0^L \big(GI_p \, \phi_{xx} + m\big) \, \delta\phi \, dx \quad (3) \quad = 0. \end{aligned}}$$

The differential equation for the rotation $\phi(x)$ of the shaft, subjected to equilibrium conditions, follows from the above relation by requiring that $(3) = 0$.

This leads to

$$\int_0^L \big(GI_p \, \phi_{xx} + m\big) \, \delta\phi \, dx = 0$$

With the fundamental Lemma of the calculus of Variations given on page 166 of the book of Török this leads to

$$\boxed{GI_p \, \phi_{xx} + m = 0}$$

b.) The boundary conditions follow from the expression on the previous page by requiring

$$① = 0,$$

i.e.,

$$\left(G I_p \, \phi_x(L) - M\right) \cdot \delta\phi(L) = 0$$

Since $\delta\phi(L) \neq 0$, this leads to the natural boundary condition

$$G I_p \, \phi_x(L) - M = 0,$$

or

$$\boxed{\phi_x(L) = \frac{M}{G I_p}}$$

Second, we need to require that

$$② = 0,$$

i.e.,

$$G I_p \, \phi_x(0) \cdot \delta\phi(0) = 0.$$

This leads to the essential boundary condition

$$\boxed{\delta\phi(0) = 0 \quad \text{or} \quad \phi(0) = 0}$$

Note that $\phi_x(0) \neq 0$, since at $x = 0$ there is a reaction moment unequal to zero.

Question (4)

a.) The kinetic energy may be written as:

$$T = \frac{1}{2} \cdot m \cdot (l \dot{\theta})^2$$
$$= \frac{1}{2} m l^2 \dot{\theta}^2.$$

The springs k are discrete, and thus have zero length. It is assumed that their orientation remains horizontal during deformation. Accordingly, the potential energy of the 2 springs follows from:

$$V^{springs} = 2 * \frac{1}{2} k \cdot (a \sin\theta)^2$$
$$= k a^2 \sin^2\theta.$$

The potential energy related to gravity is:

$$V^{gravity} = m g l \cos\theta.$$

So, the total potential energy follows as

$$V = V^{springs} + V^{gravity}$$

$$= k a^2 \sin^2\theta + m g l \cos\theta$$

The Lagrangian now becomes

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - k a^2 \sin^2\theta - m g l \cos\theta$$

b.) The Lagrangian equation of motion has the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \; ; \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m l^2 \ddot{\theta} \; ;$$

$$\frac{\partial L}{\partial \theta} = -2ka^2 \sin\theta \cos\theta + mgl \sin\theta$$
$$= -ka^2 \sin 2\theta + mgl \sin\theta$$

Accordingly, the equation of motion obtains the form:

$$\boxed{m l^2 \ddot{\theta} + ka^2 \sin 2\theta - mgl \sin\theta = 0}$$

c.) Introducing the angular velocity φ as $\varphi = \dot{\theta}$, the equation of motion above can be decomposed into two 1^{st}-order differential eqs.:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \varphi \\ \frac{-ka^2}{ml^2} \sin 2\theta + \frac{g}{l} \sin\theta \end{bmatrix}$$

Linearizing this set of differential equations about the vertical position $\theta=0$ leads to:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \left.\begin{bmatrix} 0 & 1 \\ \frac{-2ka^2}{ml^2}\cos 2\theta + \frac{g}{l}\cos\theta & 0 \end{bmatrix}\right|_{\theta=0} \begin{bmatrix} \theta \\ \varphi \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-2ka^2}{ml^2} + \frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \varphi \end{bmatrix}$$

The above system of linearized equations may be formally written as:

$$\dot{\bar{x}} = \bar{\bar{A}}\,\bar{x},$$

where we seek for solutions $\bar{x}(t) = \bar{c}\,e^{\lambda t}$ with $\bar{c}$ the amplitude vector of which the components are quantified by the initial conditions. Substituting the solution into the system of equations leads to:

$$\lambda\bar{x} = \bar{\bar{A}}\bar{x} \text{ or } (\bar{\bar{A}} - \lambda\bar{\bar{I}})\,\bar{x} = \bar{0}$$

where $\bar{\bar{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unity matrix.
The non-trivial solution now follows from:

$$|\bar{\bar{A}} - \lambda \bar{\bar{I}}| = 0 ,$$

which results in

$$\begin{vmatrix} -\lambda & 1 \\ \frac{-2ka^2}{ml^2} + \frac{g}{l} & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 + \frac{2ka^2}{ml^2} - \frac{g}{l} = 0$$

$$\Rightarrow \quad \lambda_{1,2} = \pm \sqrt{\frac{g}{l} - \frac{2ka^2}{ml^2}} .$$

The solution $\bar{x}(t) = \bar{c}\, e^{\lambda t}$ is stable if

$$\lambda \in \mathbb{C} - \mathbb{R} ,$$

which means that

$$\frac{g}{l} - \frac{2ka^2}{ml^2} < 0$$

Hence, $\boxed{k > \frac{mgl}{2a^2}}$ for the vertical position to be stable.