Elaboration of the exam "Dynamics & Stability" (AE3-914) of April 14, 2011, 18:30 - 21:30

by dr. ir. A.S.J. Suiker

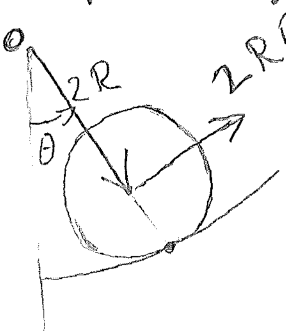
Question 1

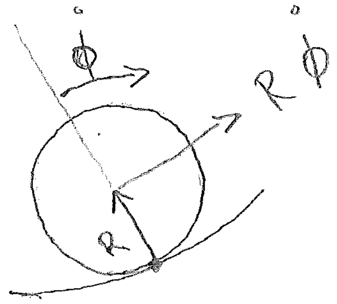
a) The translational velocity of the cylinder, $\dot{\bar{r}}_c$, will be determined from the time-derivative of the position vector of the centre of mass, $\bar{r}_c$.

$$\bar{r}_c = (z_b + 3R - 2R\cos\theta)\,\bar{K} + 2R\sin\theta\,\bar{I}$$

$$\boxed{\dot{\bar{r}}_c = (\dot{z}_b + 2R\sin\theta\,\dot{\theta})\,\bar{K} + 2R\cos\theta\,\dot{\theta}\,\bar{I}} \quad (1)$$

b) The non-holonomic constraint for $\phi(t)$ can be derived from the translational velocity of the centre of the cylinder





$$2R\dot{\theta} = R\dot{\phi} \quad (3) \quad \rightarrow \quad \boxed{\dot{\phi} = 2\dot{\theta}} \quad (4)$$

Note that the contribution of $\dot{z}_b$ is left out of (3), since it appears both in the left-hand side and the right-hand side of (3).

c.) $T = T_{block} + T_{cylinder}$ (5)

$$T_{block} = \frac{1}{2} \cdot 3m \cdot \dot{\bar{r}}_b \cdot \dot{\bar{r}}_b$$
$$= \frac{3}{2} m \cdot \dot{z}_b \bar{K} \cdot \dot{z}_b \bar{K}$$
$$= \frac{3}{2} m \dot{z}_b^2 \quad (6)$$

$$T_{cylinder} = T_{transl., cyl} + T_{rot, cyl}$$
$$= \frac{1}{2} \cdot m \cdot \dot{\bar{r}}_c \cdot \dot{\bar{r}}_c + \frac{1}{2} \cdot J_{cyl} \dot{\phi}^2 \quad (7)$$

Substituting (1) and (4) into (7), and using for the mass moment of inertia of the cylinder the expression

$$J_{cyl} = \frac{1}{2} m R^2 \quad (8)$$

leads to:

$$T_{cylinder} = \frac{1}{2} m \left(\dot{z}_b^2 + 4R \sin\theta \, \dot{z}_b \dot{\theta} + 4R^2 \dot{\theta}^2 \right) + m R^2 \dot{\theta}^2$$

$$\rightarrow T_{cylinder} = \frac{1}{2} m \left(\dot{z}_b^2 + 4R \sin\theta \, \dot{z}_b \dot{\theta} + 6R^2 \dot{\theta}^2 \right) \quad (9)$$

Inserting (6) and (9) into (5) gives for the kinetic energy of the system:

$$\boxed{T = m\left(2\dot{z}_b^2 + 2R\sin\theta\,\dot{z}_b\dot{\theta} + 3R^2\dot{\theta}^2\right)} \quad (10)$$

d.) $V = V_{block} + V_{cylinder} + V_{spring}$ (11)

$V_{block} = 3m \cdot g \cdot z_b$ (12)

$V_{cylinder} = m \cdot g\,(z_b + 3R - 2R\cos\theta)$ (13)

$V_{spring} = \frac{1}{2}k\,(z_b - z_0)^2$ (14)

Inserting (12) - (14) into (11) gives:

$$\boxed{V = 4mg z_b + mgR(3 - 2\cos\theta) + \frac{1}{2}k(z_b - z_0)^2} \quad (15)$$

e.) $L = T - V$

With (10) and (15), the lagrangian becomes:

$$L = m\left(2\dot{z}_b^2 + 2R\sin\theta\,\dot{z}_b\dot{\theta} + 3R^2\dot{\theta}^2\right) - 4mg z_b - mgR(3 - 2\cos\theta) - \frac{1}{2}k(z_b - z_0)^2 \quad (16)$$

Constructing first the equation of motion for the generalised coordinate z_b:

$$\frac{\partial L}{\partial \dot{z}_b} = 4m\dot{z}_b + 2mR\sin\theta\,\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_b}\right) = 4m\ddot{z}_b + 2mR\cos\theta\,\dot{\theta}^2 + 2mR\sin\theta\,\ddot{\theta}$$

$$\frac{\partial L}{\partial z_b} = -4mg - k(z_b - z_0)$$

Substituting these expressions into

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_b}\right) - \frac{\partial L}{\partial z_b} = 0$$

gives for the equation of motion:

$$\boxed{\begin{aligned} &4m\ddot{z}_b + 2mR\sin\theta\,\ddot{\theta} + 2mR\cos\theta\,\dot{\theta}^2 \\ &+ 4mg + k(z_b - z_0) = 0 \end{aligned}} \quad (17)$$

Constructing the equation of motion for the generalised coordinate θ:

$$\frac{\partial L}{\partial \dot{\theta}} = 2mR\sin\theta\,\dot{z}_b + 6mR^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mR\cos\theta\,\dot{\theta}\,\dot{z}_b + 2mR\sin\theta\,\ddot{z}_b + 6mR^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = 2mR\cos\theta\,\dot{z}_b\,\dot{\theta} - 2mgR\sin\theta$$

Substituting the above expressions into

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

gives for the equation of motion

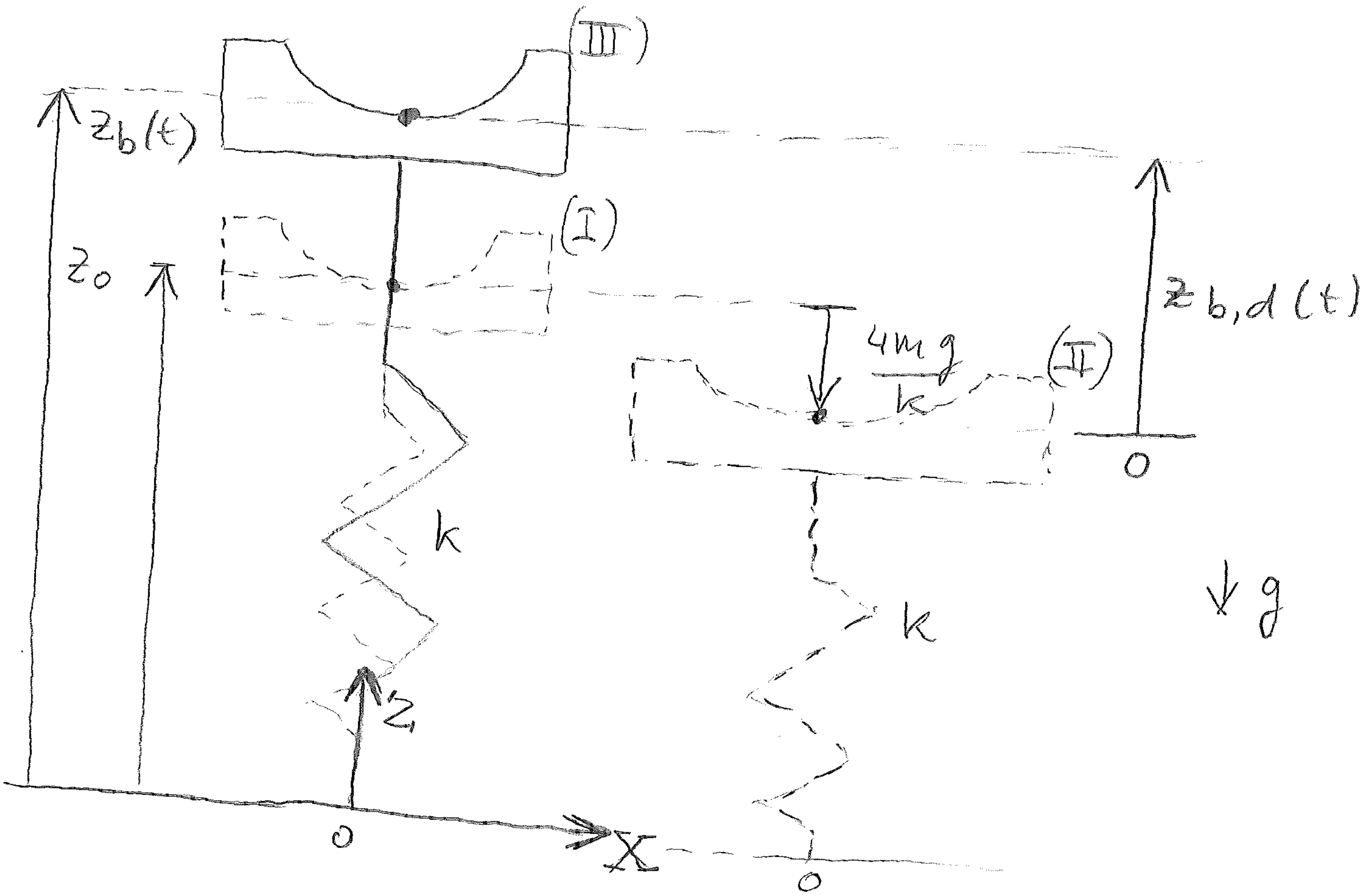
$$6mR^2\ddot{\theta} + 2mR\sin\theta\,\ddot{z}_b + 2mgR\sin\theta = 0$$

Dividing by $2mR$:

$$\boxed{3R\ddot{\theta} + \sin\theta\,\ddot{z}_b + g\sin\theta = 0} \quad (18)$$

Note: An alternative form for the equation of motion (17) can be obtained as follows:

As mentioned, the position of the centre of mass of the block equals z_0 when the spring is unstretched, i.e., before the system is subjected to a gravitational acceleration g, see Figure on the next page. When gravitation is applied, the system undergoes a static deflection of $\frac{4mg}{k}$.



(I) = position of block <u>before</u> gravity acceleration g is applied (where spring k is unstretched)

(II) = position of block <u>after</u> gravity acceleration g is applied.

(III) = position of block during dynamic motion

In correspondence with the above figure we may write:

$$z_b(t) = z_0 - \frac{4mg}{k} + z_{b,d}(t) \qquad (19)$$

where $z_{b,d}(t)$ is the position of the centre of mass of the block as measured w.r.t the <u>position of static equilibrium</u>.

Inserting (19) into (17), the equation of motion can be alternatively expressed in terms of $z_{b,d}(t)$ as :

$$\boxed{4m\ddot{z}_{b,d} + 2mR\sin\theta\,\ddot{\theta} + 2mR\cos\theta\,\dot{\theta}^2 + kz_{b,d} = 0} \quad (20)$$

The expression (20) is somewhat simpler than the expression (17), in a sense that in (20) the constant (time-independent) terms $4mg$ and kz_0 are absent.

Question (2)

a.) $\bar{\omega} = \omega\,\bar{K}$ is the rotational velocity. (1)

The inertial frame of reference X-Y-Z can be connected to the non-inertial frame of reference x-y-z via

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \\ \cos\theta & 0 & \sin\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (2)$$

From (2), the base vector $\bar{K}$ can be expressed as :

$$\bar{K} = \cos\theta \, \bar{i} + \sin\theta \, \bar{k} \qquad (3)$$

Combining (1) and (3) gives

$$\bar{\omega} = \omega \left(\cos\theta \, \bar{i} + \sin\theta \, \bar{k} \right) \qquad (4)$$

With (4), the angular momentum expressed w.r.t. the non-inertial frame of reference x-y-z becomes :

$$\boxed{\bar{L}_o = I \omega \cos\theta \, \bar{i} + J \omega \sin\theta \, \bar{k}} \qquad (5)$$

b.) For deriving the equations of motion, the time derivative $\mathring{\bar{L}}_o$ is required.

Since θ is constant, with (5) $\mathring{\bar{L}}_o$ becomes

$$\mathring{\bar{L}}_o = I \alpha \cdot \cos\theta \, \bar{i} + J \alpha \sin\theta \, \bar{k} + I \omega \cos\theta \, \mathring{\bar{i}} + J \omega \sin\theta \, \mathring{\bar{k}} \qquad (6)$$

where $\alpha = \mathring{\omega}$ is the angular acceleration (about the Z-axis).

The last two terms in equation 6 may be further developed as:

$$I\omega\cos\theta\,\dot{\bar{i}} + J\omega\sin\theta\,\dot{\bar{k}}$$

$$= I\omega\cos\theta\,(\bar{\omega}\times\bar{i}) + J\omega\sin\theta\,(\bar{\omega}\times\bar{k})$$

$$= \bar{\omega}\times(I\omega\cos\theta\,\bar{i} + J\omega\sin\theta\,\bar{k})$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega\cos\theta & 0 & \omega\sin\theta \\ I\omega\cos\theta & 0 & J\omega\sin\theta \end{vmatrix}$$

$$= \bar{j}\,(I\omega^2\sin\theta\cos\theta - J\omega^2\sin\theta\cos\theta). \quad (7)$$

Combining 6 and 7 gives:

$$\dot{\bar{L}}_0 = I\alpha\cos\theta\,\bar{i} + (I-J)\omega^2\sin\theta\cos\theta\,\bar{j} + J\alpha\sin\theta\,\bar{k} \quad (8)$$

The equations of motion in the "rotational sense" may be formally expressed as:

$$\mathring{\bar{L}}_o = \sum \bar{M}_o \qquad (9)$$

Here

$$\sum \bar{M}_o = M_x \bar{I} + M_y \bar{J} + M \bar{K} \qquad (10)$$

which, with (2), can be expressed w.r.t. the non-inertial (x-y-z) frame of reference as

$$\sum \bar{M}_o = M_x(\sin\theta\,\bar{i} - \cos\theta\,\bar{k}) + M_y\,\bar{j} + M(\cos\theta\,\bar{i} + \sin\theta\,\bar{k}) \qquad (11)$$

Inserting (8) and (11) into (9) gives

$$\boxed{\begin{aligned} &I\alpha\cos\theta\,\bar{i} + (I-J)\,\omega^2\sin\theta\cos\theta\,\bar{j} + J\alpha\sin\theta\,\bar{k} \\ &= (M_x\sin\theta + M\cdot\cos\theta)\,\bar{i} + M_y\,\bar{j} \\ &\quad -(M_x\cos\theta - M\sin\theta)\,\bar{k} \end{aligned}} \qquad (12)$$

In terms of components, Eq. (12) is expressed as:

$$\boxed{\begin{aligned} I\alpha\cos\theta &= M_x\sin\theta + M\cos\theta && (13)\\ (I-J)\,\omega^2\sin\theta\cos\theta &= M_y && (14)\\ J\alpha\sin\theta &= -M_x\cos\theta + M\sin\theta && (15) \end{aligned}}$$

c.) Expressing both (13) and (15) explicitly in terms of M_x, and equating the two expressions gives:

$$\frac{I\alpha\cos\theta - M\cos\theta}{\sin\theta} = \frac{J\alpha\sin\theta - M\sin\theta}{-\cos\theta}$$

$$\Rightarrow \quad M(\cos^2\theta + \sin^2\theta) = J\alpha\sin^2\theta + I\alpha\cos^2\theta$$

$$\Rightarrow \quad \boxed{\alpha = \frac{M}{J\sin^2\theta + I\cos^2\theta}} \quad (16)$$

Next, inserting (16) into (13) leads to:

$$\frac{IM\cdot\cos\theta}{J\sin^2\theta + I\cos^2\theta} = M_x\sin\theta + M\cos\theta$$

$$\Rightarrow \quad M\cos\theta\left[\frac{I}{J\sin^2\theta + I\cos^2\theta} - \frac{J\sin^2\theta + I\cos^2\theta}{J\sin^2\theta + I\cos^2\theta}\right]$$

$$= M_x\sin\theta$$

$$\boxed{M_x = \frac{(I - J)\sin\theta\cos\theta\, M}{J\sin^2\theta + I\cos^2\theta}} \quad (17)$$

Question (3)

a) $$T = \int_0^L \frac{1}{2}\rho \cdot y_t^2 \, dx \quad (1)$$ is the kinetic energy,

where $$y_t = \frac{\partial y(x,t)}{\partial t}$$

The Lagrangian is formulated as

$$L = T - V \quad (2)$$

which, with the kinetic energy given by (1) and the energy potential V as

$$V = \int_0^L \left(\frac{1}{2} T y_x^2 - q y\right) dx \quad (3)$$

becomes

$$L = \int_0^L \left(\frac{1}{2}\rho y_t^2 - \frac{1}{2} T y_x^2 + q y\right) dx \quad (4)$$

Substituting the Lagrangian (4) into Hamilton's principle,

$$\delta \int_{t_0}^{t_1} L \, dt = 0 \qquad (5)$$

leads to:

$$\int_{t_0}^{t_1} \int_{0}^{L} \left(\rho y_t \, \delta y_t - T y_x \, \delta y_x + q \, \delta y \right) dx \, dt = 0 \qquad (6)$$

Using integration by parts for the first and second terms gives:

$$\int_{0}^{L} \rho y_t \, \delta y \Big|_{t_0}^{t_1} dx - \int_{t_0}^{t_1} \int_{0}^{L} \rho u_{tt} \, \delta y \, dx \, dt$$

$$- \int_{t_0}^{t_1} T y_x \, \delta y \Big|_{0}^{L} dt + \int_{t_0}^{t_1} \int_{0}^{L} T y_{xx} \, \delta y \, dx \, dt$$

$$+ \int_{t_0}^{t_1} \int_{0}^{L} q \, \delta y \, dx \, dt = 0 \qquad (7)$$

Reordering this expression leads to

$$\int_{t_0}^{t_1} \int_{0}^{L} \underbrace{\left(T y_{xx} - \rho y_{tt} + q \right)}_{(1)} \delta y \, dx \, dt$$

$$+ \underbrace{\int_{0}^{L} \rho y_t \, \delta y \Big|_{t_0}^{t_1} dx}_{(2)} - \underbrace{\int_{t_0}^{t_1} T y_x \, \delta y \Big|_{0}^{L} dt}_{(3)} = 0 \qquad (8)$$

From 8, the equation of motion follows as

1 $= 0$: $\boxed{T y_{xx} - \rho y_{tt} + q = 0 \quad \text{for } 0<x<L}$ 9

b.) From 8, the boundary conditions follow as:

3 $= 0$: $T y_x \, \delta y \Big|_0^L = 0$

$\Rightarrow \quad \delta y(L,t) = 0 \quad \forall t \quad$ since $\quad y(L,t) = 0 \quad \forall t$ 10

$\Rightarrow \quad \delta y(0,t) = 0 \quad \forall t \quad$ since $\quad y(0,t) = 0 \quad \forall t$ 11

Both 10 and 11 are essential boundary conditions.

Question 4

a.) The potential energy of the system has the form:

$$V = V_{pendulums} + V_{block} + V_{spring} \quad (1)$$

Here,

$$V_{pendulums} = 2 * \left(mg \cdot \tfrac{1}{2} l \cos\theta \right) \quad (2)$$

$$V_{block} = 4m \cdot g \cdot \left(l\cos\theta + \tfrac{1}{2} h \right) \quad (3)$$

$$V_{spring} = \tfrac{1}{2} k \left(l \sin\theta \right)^2 \quad (4)$$

Substituting (2) - (4) into (1) gives for the potential energy of the system

$$V = 5mgl\cos\theta + 2mgh + \tfrac{1}{2} k \left(l\sin\theta \right)^2 \quad (5)$$

Subsequently an effective potential V_{eff} is established as:

$$V_{eff} = V + V_{force} \quad (6)$$

The contribution V_{force} is calculated from the relation

$$\delta V_{force} = -\delta W \quad (7)$$

with the virtual work as

$$\delta W = -F \cdot \delta z \quad (8)$$

Since the vertical position of the load equals

$$z = l\cos\theta + h \quad (9)$$

we obtain that

$$\delta z = -l \sin\theta \, \delta\theta \quad (10)$$

Combining (8) and (10) with (7) gives

$$\delta V_{force} = -Fl \sin\theta \, \delta\theta \quad (11)$$

from which we obtain

$$V_{force} = Fl \cos\theta \quad (12)$$

Inserting (5) and (12) into (6) leads to

$$V_{eff} = 5mgl\cos\theta + 2mgh + \tfrac{1}{2}kl^2\sin^2\theta + Fl\cos\theta \quad (13)$$

Furthermore, the kinetic energy is formulated as:

$$T = T_{pendulums} + T_{block} \quad (14)$$

When formulating the kinetic energy w.r.t. the point of rotation of the pendulums, we obtain

$$T_{pendulums} = 2 \cdot T_{rot,pendulum}$$

$$= 2 * \left(\tfrac{1}{2} J_o \cdot \dot{\theta}^2\right)$$

$$= \tfrac{1}{3} m l^2 \dot{\theta}^2 \quad (15)$$

For computing the kinetic energy of the block, we first formulate the centre of mass $\bar{r}_b$ as

$$\bar{r}_b = -l\sin\theta\,\bar{I} + \left(l\cos\theta + \tfrac{1}{2}h\right)\bar{J} \qquad (16)$$

The velocity of the centre of mass then becomes

$$\dot{\bar{r}}_b = -l\cos\theta\,\dot{\theta}\,\bar{I} - l\sin\theta\,\dot{\theta}\,\bar{J} \qquad (17)$$

Since the block does not rotate, with (17) the kinetic energy of the block becomes

$$T_{block} = \tfrac{1}{2}\cdot 4m\,\dot{\bar{r}}_b\cdot\dot{\bar{r}}_b$$

$$= 2ml^2\dot{\theta}^2 \qquad (18)$$

Inserting (15) and (18) into (14) leads to

$$T = \tfrac{7}{3}ml^2\dot{\theta}^2 \qquad (19)$$

The Lagrangian of the system is written as

$$L = T - V_{eff} \qquad (20)$$

which, with (19) and (13) becomes:

$$\boxed{L = \frac{7}{3} m l^2 \dot{\theta}^2 - 5mgl\cos\theta - 2mgh - \frac{1}{2} k l^2 \sin^2\theta - Fl\cos\theta} \quad (21)$$

Note: The constant term $2mgh$ may be left out of the Lagrangian, since the Lagrangian (or potential energy) is unique up to a constant (which reflects the choice of the datum used for formulating the potential energy).

b.) The equation of motion is expressed as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad (22)$$

Using (21), we obtain:

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{14}{3} m l^2 \dot{\theta} \; ; \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{14}{3} m l^2 \ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = 5mgl\sin\theta - kl^2 \sin\theta\cos\theta + Fl\sin\theta$$

Substituting these derivatives into (22) gives for the equation of motion:

$$\boxed{\frac{14}{3} m l^2 \ddot{\theta} + (kl\cos\theta - 5mg - F)\, l \sin\theta = 0}$$

(23)

c.) Introducing $\dot{\theta} = \varphi$, the above equation of motion may be written in terms of two 1^{st} order differential equations:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \varphi \\ \dfrac{-3(kl\cos\theta - 5mg - F)\sin\theta}{14\, m l} \end{bmatrix}$$

Linearizing this set of differential equations about the vertical position $\theta = 0$ leads to

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \left.\begin{bmatrix} 0 & 1 \\ \dfrac{-3kl\cos 2\theta + 3\cos\theta(5mg + F)}{14\, m l} & 0 \end{bmatrix}\right|_{\theta=0} \begin{bmatrix} \theta \\ \varphi \end{bmatrix}$$

which thus may be written as

$$\begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-3kl + 15mg + 3F}{14\,ml} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \varphi \end{bmatrix} \quad (24)$$

This system of equations may be formally written as :

$$\dot{\bar{x}} = \bar{\bar{A}}\,\bar{x} \quad (25)$$

where we seek for solutions $\bar{x}(t) = \bar{c}\,e^{\lambda t}$, with $\bar{c}$ the amplitude vector of which the components are determined by the initial conditions. Substituting this solution into (25) leads to :

$$\lambda\bar{x} = \bar{\bar{A}}\,\bar{x} \Rightarrow \left(\bar{\bar{A}} - \lambda\bar{\bar{I}}\right)\bar{x} = 0, \quad (26)$$

where $\bar{\bar{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the unity matrix.

The non-trivial solution of (26) follows as

$$\left|\bar{\bar{A}} - \lambda\bar{\bar{I}}\right| = 0,$$

which results in :

$$\begin{vmatrix} -\lambda & 1 \\ \dfrac{-3kl + 15mg + 3F}{14ml} & -\lambda \end{vmatrix} = 0 \quad (26^A)$$

$$\Rightarrow \quad \lambda^2 + \frac{3k}{14m} - \frac{15g}{14l} - \frac{3F}{14ml} = 0$$

$$\Rightarrow \quad \lambda_{1,2} = \pm\sqrt{\frac{3F}{14ml} + \frac{15g}{14l} - \frac{3k}{14m}} \quad (27)$$

The solution $\bar{x}(t) = \bar{c}\, e^{\lambda t}$ is stable if $\lambda \in \mathbb{C} - \mathbb{R}$,

which means that

$$\frac{3F}{14ml} + \frac{15g}{14l} - \frac{3k}{14m} < 0$$

Hence, the stability requirement for F becomes:

$$\boxed{F < kl - 5mg} \quad (28)$$

An <u>alternative procedure</u> for examining the stability of the system is to use the effective potential V_{eff} given by (13).

From this expression, it can be computed that

$$\frac{dV_{eff}}{d\theta} = -5mgl\sin\theta + kl^2\sin\theta\cos\theta - Fl\sin\theta . \quad (29)$$

Indeed, from (29) it follows that the position $\theta = 0$ provides a state of equilibrium,

$$\left.\frac{dV_{eff}}{d\theta}\right|_{\theta=0} = 0 .$$

Subsequently, from (29) we obtain

$$\frac{d^2V_{eff}}{d\theta^2} = -5mgl\cos\theta + kl^2(\cos^2\theta - \sin^2\theta) - Fl\cos\theta ,$$

from which we obtain that

$$\left.\frac{d^2V_{eff}}{d\theta^2}\right|_{\theta=0} = -5mgl + kl^2 - Fl \quad (30)$$

The requirement for stability is that

$$\left.\frac{d^2 V_{eff}}{d\theta^2}\right|_{\theta=0} > 0 \qquad 30^A$$

With 30, this leads to

$$-5mgl + kl^2 - Fl > 0$$

$$\Rightarrow \boxed{F < kl - 5mg} \qquad 31$$

which is indeed the same requirement as 28.