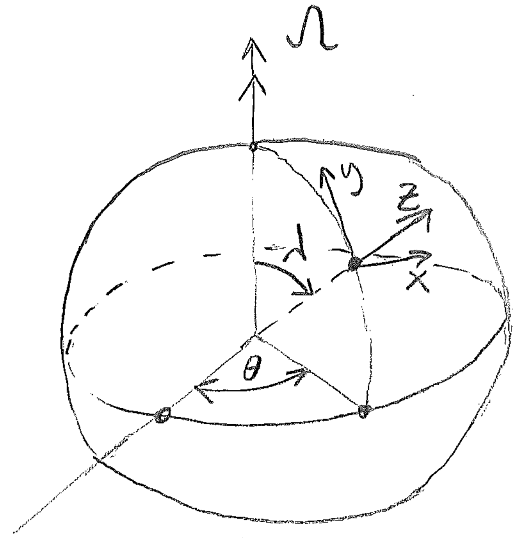


Fig I

↑ see above Figure

λ = colatitude angle, which is $90°$ – latitude angle.
The rotation of the earth is given by $\bar{\Omega}$.

1.167 an object is thrown vertically upward with a speed V_0. Prove that when it returns it will be at a distance westward from its starting point equal to $(4\Omega V_0^3 \sin\lambda)/3g^2$.

Note: For deriving this expression the terms relating to Ω^2 are neglected ($\Omega^2 \simeq 0$).

In the non-inertial frame of reference x-y-z Newton's 2nd law is formulated as

$$m \cdot \bar{a}_{rel} = \bar{F} + \bar{F}_{fict} \qquad (1)$$

with the fictitious force given by

$$\bar{F}_{fict} = -m\left(\bar{a}_{xyz} + \dot{\bar{\omega}}_{xyz} \times \bar{r}_{rel} + \bar{\omega}_{xyz} \times (\bar{\omega}_{xyz} \times \bar{r}_{rel}) + 2\,\bar{\omega}_{xyz} \times \bar{v}_{rel}\right) \qquad (2)$$

In this expression,

$$\bar{a}_{xyz} = \bar{\omega} \times (\bar{\omega} \times \bar{r}_{xyz}) \quad (3)$$

with $\bar{r}_{xyz}$ the distance from the centre of the earth to the origin of the rotating x-y-z frame of reference.

Further, $\bar{\omega}_{xyz}$ is related to the earth's rotation as

$$\bar{\omega}_{xyz} = \Omega\left(\sin\lambda\,\bar{j} + \cos\lambda\,\bar{k}\right) \quad (4)$$

$$\dot{\bar{\omega}}_{xyz} = \bar{0}, \text{ (since } \Omega \text{ is constant).} \quad (5)$$

$$\left.\begin{aligned} \bar{r}_{rel} &= x\,\bar{i} + y\,\bar{j} + z\,\bar{k} \\ \bar{v}_{rel} &= \dot{x}\,\bar{i} + \dot{y}\,\bar{j} + \dot{z}\,\bar{k} \\ \bar{a}_{rel} &= \ddot{x}\,\bar{i} + \ddot{y}\,\bar{j} + \ddot{z}\,\bar{k} \end{aligned}\right\} \quad (6)$$

Substituting (4) into (3) gives

$$\bar{a}_{xyz} = \Omega(\sin\lambda\,\bar{j} + \cos\lambda\,\bar{k}) \times \left[\Omega(\sin\lambda\,\bar{j} + \cos\lambda\,\bar{k}) \times (x\bar{i} + y\bar{j} + z\bar{k})\right]$$

$$= \Omega^2(\sin\lambda\,\bar{j} + \cos\lambda\,\bar{k}) \times \left[(\sin\lambda\,\bar{j} + \cos\lambda\,\bar{k}) \times (x\bar{i} + y\bar{j} + z\bar{k})\right]$$

Since this term is proportional to Ω^2, it is ignored as stated by $\Omega^2 \simeq 0$;

$$\bar{a}_{xyz} \simeq \bar{0}. \quad (7)$$

Using the same argument, the term

$\bar{\omega}_{xyz} \times (\bar{\omega}_{xyz} \times \bar{r}_{REL})$, which also is proportional to Ω^2, is ignored.

Hence, Eq. 2 is reduced to an expression that only contains the Coriolis ACCELERATION:

$$\bar{F}_{fict} = -m \cdot 2\,\bar{\omega}_{xyz} \times V_{REL}$$

$$= -2m\Omega(\sin\lambda\,\bar{j} + \cos\lambda\,\bar{k}) \times (\dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k})$$

$$= -2m\Omega\left(-\dot{x}\sin\lambda\,\bar{k} + \dot{z}\sin\lambda\,\bar{i} + \dot{x}\cos\lambda\,\bar{j} - \dot{y}\cos\lambda\,\bar{i}\right) \qquad (8)$$

Inserting 8, 6_{-3} into 1, and using the fact that the External loading $\bar{F}$ is defined as

$$\bar{F} = -m \cdot g\,\bar{k} \qquad (9)$$

gives

$$m(\ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}) = -mg\bar{k} - 2m\Omega\left[(\dot{z}\sin\lambda - \dot{y}\cos\lambda)\bar{i} + \dot{x}\cos\lambda\,\bar{j} - \dot{x}\sin\lambda\,\bar{k}\right] \qquad (10)$$

Factorizing m out, and separating the equations of motion for the x, y and z-direction gives:

$$\ddot{x} + 2\Omega \sin\alpha \, \dot{z} - 2\Omega \cos\alpha \, \dot{y} = 0; \quad (11)$$

$$\ddot{y} + 2\Omega \cos\alpha \, \dot{x} = 0; \quad (12)$$

$$\ddot{z} = 2\Omega \sin\alpha \, \dot{x} - g; \quad (13)$$

Integrating the above expressions w.r.t. time gives, respectively;

$$\dot{x} + 2\Omega \sin\alpha \, z - 2\Omega \cos\alpha \, y = C_1 \quad (14)$$

$$\dot{y} + 2\Omega \cos\alpha \, x = C_2 \quad (15)$$

$$\dot{z} = 2\Omega \sin\alpha \, x - gt + C_3 \quad (16)$$

The integration constants C_1, C_2 and C_3 follow from the initial conditions:

$x(0) = y(0) = z(0) = 0$; $\dot{x}(0) = 0$; $\dot{y}(0) = 0$; $\dot{z}(0) = V_0$

After substituting these intial contions into (14), (15) and (16), we obtain $C_1 = 0$; $C_2 = 0$; $C_3 = V_0$, so

$$\dot{x} + 2\Omega \sin\alpha \, z - 2\Omega \cos\alpha \, y = 0 \quad (17)$$

$$\dot{y} + 2\Omega \cos\alpha \, x = 0 \quad (18)$$

$$\dot{z} = 2\Omega \sin\alpha \, x - gt + V_0 \quad (19)$$

Inserting 17 - 19 into 11 - 13, where the terms relating to Ω^2 are ignored gives the simplified equations of motion:

$$\ddot{x} + 2\Omega \sin\lambda \, (-gt + V_0) = 0 \quad (20)$$

$$\ddot{y} = 0 \quad (21)$$

$$\ddot{z} = -g \quad (22)$$

Integrating the above expression twice w.r.t. time, again using the initial conditions $x(0) = y(0) = z(0) = 0$; $\dot{x}(0) = 0$; $\dot{y}(0) = 0$; $\dot{z}(0) = V_0$; leads t

$$x(t) = \frac{1}{3}\Omega \sin\lambda \, g t^3 - \Omega \sin\lambda \, V_0 t^2 \quad (23)$$

$$y(t) = 0 \quad (24)$$

$$z(t) = V_0 t - \frac{1}{2} g t^2 \quad (25)$$

The time it takes for the object to come back to the earth ($z=0$) after being thrown up can be computed from Eq. 25 :

$$0 = V_0 t - \frac{1}{2} g t^2$$

$$\rightarrow t(V_0 - \frac{1}{2} g t) = 0$$

$$\rightarrow t = 0 \quad \vee \quad t = \frac{2V_0}{g}$$

Substituting $t = \frac{2V_0}{g}$ into (23) gives

$$\boxed{x\left(\frac{2V_0}{g}\right) = \frac{1}{3}\Omega \sin\lambda \, g \left(\frac{2V_0}{g}\right)^3 - \Omega \sin\lambda \, V_0 \left(\frac{2V_0}{g}\right)^2 = -\frac{4}{3}\Omega \sin\lambda \frac{V_0^3}{g^2}} \quad (26)$$

which means that is has indeed moved over a distance

$$\frac{4}{3}\Omega \sin\lambda \frac{V_0^3}{g^2}$$

in the westward direction (in the negative x-direction, see Figure I)

1.169

The solution to this problem essentially uses expressions derived for question 1.167.

The projectile now is not thrown vertically but under an angle α with x-axis in the west direction. So, the initial conditions are now:

$$x(0) = y(0) = z(0) = 0 \quad (27)$$

$$\dot{x}(0) = -V_0 \cdot \cos\alpha \; ; \; \dot{y}(0) = 0 \; ; \; \dot{z}(0) = V_0 \sin\alpha$$

Substituting the boundary conditions (27) into (14) - (16) gives

$$-V_0 \cos\alpha = C_1$$

$$0 = C_2$$

$$V_0 \sin\alpha = C_3$$

This turns the expressions (14) - (16) into

$$\dot{x} + 2\Omega \sin\lambda\, z - 2\Omega \cos\lambda\, y = -V_0 \cos\alpha \quad (28)$$

$$\dot{y} + 2\Omega \cos\lambda\, x = 0 \quad (29)$$

$$\dot{z} = 2\Omega \sin\lambda\, x - gt + V_0 \sin\alpha \quad (30)$$

Inserting (28) - (30) into (11) - (13) and ignoring the terms related to Ω^2 provides the <u>simplified</u> Eqs. of motion

$$\ddot{x} + 2\Omega \sin\lambda\,(-gt + V_0 \sin\alpha) = 0; \quad (31)$$

$$\ddot{y} + 2\Omega \cos\lambda\,(-V_0 \cos\alpha) = 0; \quad (32)$$

$$\ddot{z} = 2\Omega \sin\lambda\,(-V_0 \cos\alpha) - g \quad (33)$$

Integrating the above expressions once w.r.t. time gives

$$\dot{x} = \Omega \sin\lambda \, g t^2 - 2\Omega \sin\lambda \, V_0 \sin\alpha \, t + C_1 \quad (34)$$

$$\dot{y} = 2\Omega \cos\lambda \, V_0 \cos\alpha \, t + C_2 \quad (35)$$

$$\dot{z} = -2\Omega \sin\lambda \cos\alpha \, V_0 t - g t + C_3 \quad (36)$$

The integration constants C_1, C_2 and C_3 follow from the initial conditions (27) as

$$C_1 = -V_0 \cos\alpha$$

$$C_2 = 0 \quad (37)$$

$$C_3 = V_0 \sin\alpha$$

Integrating (34)-(36) once again w.r.t. time, and using (37) leads to

$$x = \frac{1}{3} \Omega \sin\lambda \, g t^3 - \Omega \sin\lambda \, V_0 \sin\alpha \, t^2 - V_0 \cos\alpha t$$

$$y = \Omega \cos\lambda \, V_0 \cos\alpha \, t^2 \quad (38)$$

$$z = -\Omega \sin\lambda \cos\alpha \, V_0 t^2 - \frac{1}{2} g t^2 + V_0 \sin\alpha t$$

where the integration constants are zero as a result of the initial conditions, Eq (27)

The projectile reaches the maximum height when $\dot{z}(t) = 0$.

From 36-3 and 37-3, this gives

$$0 = -2\Omega \sin\lambda \cos\alpha \, V_0 t - gt + V_0 \sin\alpha$$

$$\Rightarrow t = \frac{V_0 \sin\alpha}{2 V_0 \Omega \sin\lambda \cos\alpha + g} \quad (39)$$

which can be further developed as

$$t = \frac{V_0 \sin\alpha}{2 V_0 \Omega \sin\lambda \cos\alpha + g} * \frac{-2 V_0 \Omega \sin\lambda \cos\alpha + g}{-2 V_0 \Omega \sin\lambda \cos\alpha + g}$$

$$\rightarrow t = \frac{V_0 \sin\alpha \, (g - 2 V_0 \Omega \sin\lambda \cos\alpha)}{g^2 - \underbrace{4 V_0^2 \Omega^2 \sin^2\lambda \cos^2\alpha}_{\simeq 0 \text{ since } \Omega^2 \simeq 0}}$$

$$\rightarrow \boxed{t \simeq \frac{V_0 \sin\alpha \, (g - 2 V_0 \Omega \sin\lambda \cos\alpha)}{g^2}} \quad (40)$$

When $\Omega = 0$, (40) turns into

$$\boxed{t \simeq \frac{V_0 \sin\alpha}{g}}$$

(simplification, assuming the earth does not rotate).

1.170 The maximum height can be computed (1) by substituting (40) into (30)_3:

2

$$z_{max} = \left(-\frac{1}{2}g - \Omega \sin\lambda \cos\alpha\, V_0\right) * \left[\frac{V_0 \sin\alpha\,(g - 2V_0\Omega \sin\lambda \cos\alpha)}{g^2}\right]^2$$

$$+ V_0 \sin\alpha \left[\frac{V_0 \sin\alpha\,(g - 2V_0 \Omega \sin\lambda \cos\alpha)}{g^2}\right]$$

Ignoring the terms related to Ω^2, this expression can be developed as

$$z_{max} = -\frac{1}{2}g\,\frac{V_0^2 \sin^2\alpha\, g^2}{g^4}$$

$$- \Omega \sin\lambda \cos\alpha\, V_0 \cdot \frac{V_0^2 \sin^2\alpha\, g^2}{g^4}$$

$$- \frac{4V_0 \Omega \sin\lambda \cos\alpha\, g\, V_0^2 \sin^2\alpha}{g^4} * \left(-\frac{1}{2}g\right)$$

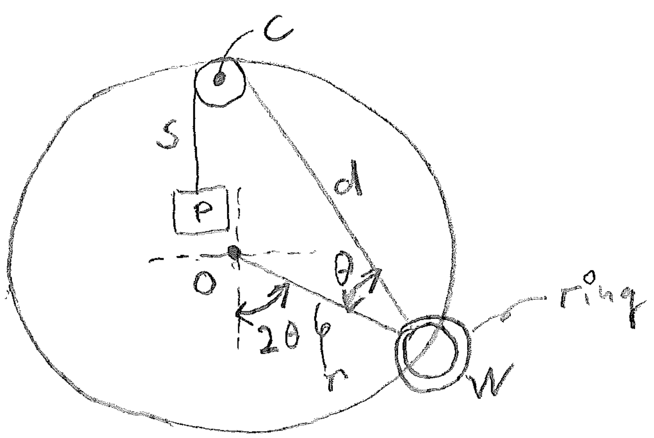
$$+ \frac{V_0^2 \sin^2\alpha\, g}{g^2} - \frac{2V_0^3 \Omega \sin^2\alpha \sin\lambda \cos\alpha}{g^2}$$

which finally leads to:

$$\boxed{z_{max} = \frac{V_0^2 \sin^2\alpha}{2g} - \frac{V_0^3 \Omega \sin^2\alpha \sin\lambda \cos\alpha}{g^2}}$$

NOTE: The factor of 2 appearing in front of the second term on page 73 of Tövik is wrong!

1.207



Let s be the length of the string attached to the block and let d be the length of the string between the pulley and the ring.

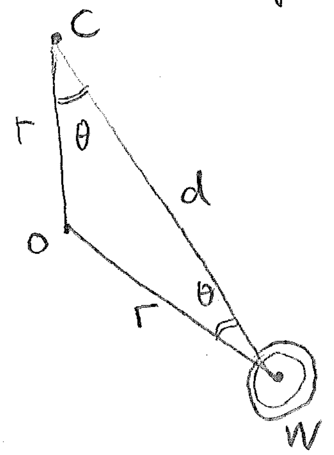
$$s + d = l \ (= \text{constant}) \quad (1)$$

with l the total length of the string.

When writing the angle 2θ as $\phi = 2\theta$, and giving the ring a virtual displacement, the virtual work can be written as

$$\delta W = P . \delta s - W . r \sin\phi \ \delta\phi \quad (2)$$

The distance d follows, with the triangle below



as $d = 2r\cos\theta$, from which it follows that

$$\delta d = -2r\sin\theta \ \delta\theta \quad (3)$$

Further, from 1 it follows that

$$\delta s + \delta d = 0, \quad \text{so that} \quad \delta s = -\delta d \quad (4)$$

In addition, since $\phi = 2\theta$, we have

$$\delta\phi = 2\delta\theta \quad (5)$$

Substituting 3 - 5 into 2 gives

$$\delta W = P \cdot 2r\sin\theta\,\delta\theta - 2Wr\sin 2\theta\,\delta\theta. \quad (6)$$

For equilibrium, it is required that $\delta W = 0$, so, from 6 it follows that

$$(P2r\sin\theta - 2Wr\sin 2\theta)\,\delta\theta = 0 \quad (7)$$

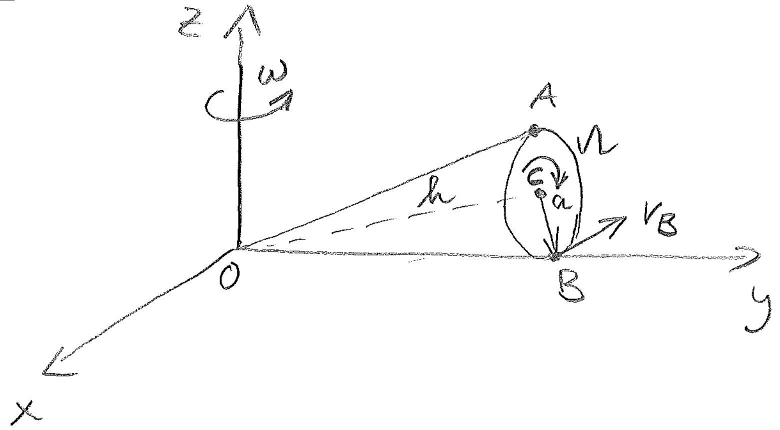
which, for an arbitrary $\delta\theta$, leads to

$$P2r\sin\theta - 2Wr\sin 2\theta = 0$$

$$\rightarrow \quad P\cancel{\sin}\theta - W \cdot 2\cancel{\sin}\theta\cos\theta = 0$$

$$\rightarrow \quad \boxed{P = 2W\cos\theta}$$

4.4.0



The contact point B of the line of contact OB has a velocity equal to

$$\vec{V}_B = \bar{\omega} \times \vec{OB}$$

$$= \omega \bar{k} \times \sqrt{h^2+a^2} \cdot \bar{j}$$

$$= -\omega \sqrt{h^2+a^2} \cdot \bar{i} \qquad ①$$

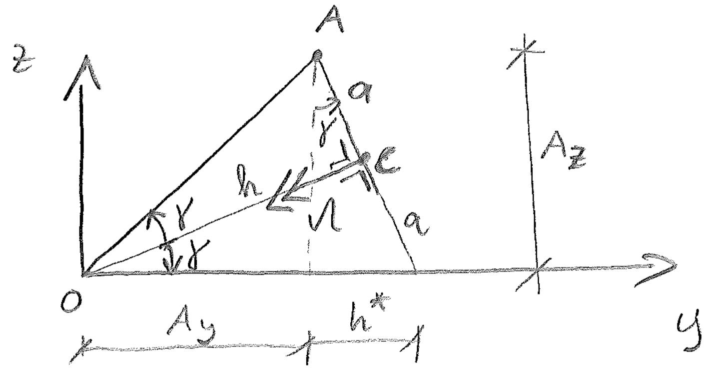
This velocity may be alternatively written in terms of the angular velocity Ω of the cone as

$$\vec{V}_B = -\Omega a \bar{i} \qquad ②$$

Equating ① and ② gives

$$\boxed{\Omega = \frac{\omega \sqrt{h^2+a^2}}{a}} \qquad ③ \qquad (= \text{holonomic constraint !})$$

When denoting the centre point of the base of the cone as C (see Figure above), the total rotation of a point on the cone is determined by the rotation ω about the z-axis, and the rotation Ω about the OC-axis.



In accordance with the above figure, the rotation $\bar{\Omega}$ about the OC axis can be written as :

$$\bar{\Omega} = \Omega_y \cdot \bar{j} + \Omega_z \bar{k}$$

$$= -\Omega \cos\gamma \, \bar{j} - \Omega \sin\gamma \, \bar{k}$$

$$= -\Omega \frac{h}{\sqrt{h^2+a^2}} \bar{j} - \Omega \frac{a}{\sqrt{h^2+a^2}} \bar{k}$$

The total rotation $\bar{\omega}$ then follows as

$$\bar{\omega} = \bar{\Omega} + \omega \bar{k}$$

$$= -\Omega \frac{h}{\sqrt{h^2+a^2}} \bar{j} + \left(\omega - \frac{\Omega a}{\sqrt{h^2+a^2}}\right) \bar{k} \qquad (4)$$

The velocity of the highest point on the cone, point A, can now be computed as :

$$\vec{V}_A = \vec{\omega} \times \vec{OA} \qquad (5)$$

where $\vec{OA}$ follows from

$$\vec{OA} = A_y \cdot \bar{j} + A_z \cdot \bar{k}$$

$$= \sqrt{h^2+a^2} \cdot \cos 2\gamma \cdot \bar{j} + \sqrt{h^2+a^2} \cdot \sin 2\gamma \, \bar{k}$$

HENCE $\vec{OA}$ can be further elaborated as

$$\vec{OA} = \sqrt{h^2+a^2}\,(\cos^2\gamma - \sin^2\gamma)\,\bar{j} + \sqrt{h^2+a^2}\cdot 2\sin\gamma\cdot\cos\gamma\,\bar{k}$$

$$\to \vec{OA} = \sqrt{h^2+a^2}\left(\left[\left(\frac{h}{\sqrt{h^2+a^2}}\right)^2 - \left(\frac{a}{\sqrt{h^2+a^2}}\right)^2\right]\bar{j} + 2\cdot\frac{h}{\sqrt{h^2+a^2}}\cdot\frac{a}{\sqrt{h^2+a^2}}\,\bar{k}\right)$$

$$\to \vec{OA} = \frac{h^2-a^2}{\sqrt{h^2+a^2}}\,\bar{j} + \frac{2ha}{\sqrt{h^2+a^2}}\,\bar{k} \quad (6)$$

Now, substituting (6) and (4) into (5) gives:

$$\vec{V}_A = \left(-\Omega\frac{h}{\sqrt{h^2+a^2}}\,\bar{j} + \left(\omega - \frac{\Omega a}{\sqrt{h^2+a^2}}\right)\bar{k}\right) \times \left(\frac{h^2-a^2}{\sqrt{h^2+a^2}}\,\bar{j} + \frac{2ha}{\sqrt{h^2+a^2}}\,\bar{k}\right)$$

$$\to \vec{V}_A = -\Omega\frac{h}{\sqrt{h^2+a^2}}\cdot\frac{2ha}{\sqrt{h^2+a^2}}\cdot\bar{i} - \left(\omega - \frac{\Omega a}{\sqrt{h^2+a^2}}\right)\cdot\frac{(h^2-a^2)}{\sqrt{h^2+a^2}}\,\bar{i}$$

$$\to \vec{V}_A = \left[-\Omega\left\{\frac{h^2a}{h^2+a^2} + \frac{a^3}{h^2+a^2}\right\} - \omega\frac{(h^2-a^2)}{\sqrt{h^2+a^2}}\right]\bar{i} \quad (7)$$

Inserting 3 into 7 gives

$$\vec{V}_A = \left[-\frac{\omega\sqrt{h^2+a^2}}{a} \left\{ \frac{h^2 a}{h^2+a^2} + \frac{a^3}{h^2+a^2} \right\} - \omega \frac{(h^2-a^2)}{\sqrt{h^2+a^2}} \right] \bar{i}$$

$$\rightarrow \boxed{\vec{V}_A = \frac{-2 \cdot h^2 \cdot \omega}{\sqrt{h^2+a^2}} \bar{i}}$$