2.17 ①

The kinetic energy of the system Figure ① is

$$T = \frac{1}{2} J \dot{\theta}^2$$
$$= \frac{1}{2} \cdot m \cdot (2a)^2 \cdot \dot{\theta}^2$$
$$= 2 m a^2 \dot{\theta}^2 \quad ①$$

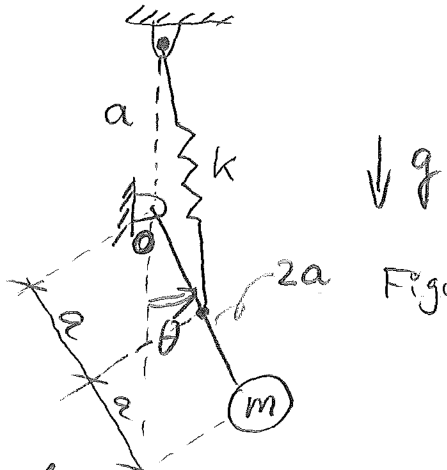


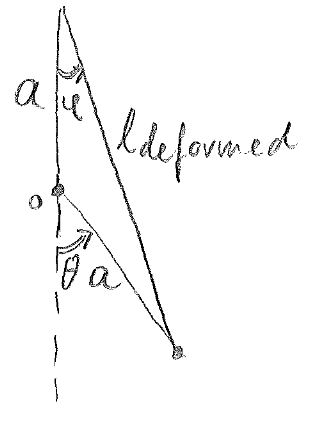
Figure ①

The potential energy of the system is

$$V = -mg \cdot 2a\cos\theta + \frac{1}{2} k \cdot \delta_{spring}^2 \quad ②$$

$$\delta_{spring} = l_{deformed} - l \quad ③$$

with $l_{deformed}$ as given in the figure below




$$l_{deformed} = 2a\cos\varphi \quad ④$$

$\varphi$ can be expressed in terms of $\theta$ from the relation;

$$a + a\cos\theta = l_{deformed}\cos\varphi$$
⑤

Substituting 4 into 5 gives

$$a + a\cos\theta = 2a\cos^2\varphi$$

$$\Rightarrow \quad \cos\varphi = \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos\theta} \qquad (6)$$

Inserting 6 into 4 leads to

$$l_{deformed} = 2a\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\cos\theta}$$

$$= a\sqrt{2+2\cos\theta} \qquad (7)$$

With this expression, 3 becomes

$$\delta_{spring} = a\sqrt{2+2\cos\theta} - \ell \qquad (8)$$

Inserting 8 into 2 leads to

$$V = -2mga\cos\theta + \tfrac{1}{2}k\left(a\sqrt{2+2\cos\theta} - \ell\right)^2 \qquad (9)$$

From 1 and 9, the Lagrangian $L = T - V$ becomes

$$L = 2ma^2\dot{\theta}^2 + 2mga\cos\theta - \tfrac{1}{2}k\left(a\sqrt{2+2\cos\theta} - \ell\right)^2 \qquad (10)$$

Substituting 10 into the Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

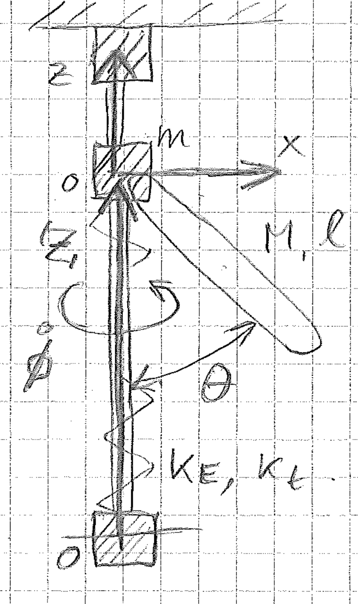
leads to

$$\boxed{4ma^2\ddot{\theta} + 2mga\sin\theta - k\left(a\sqrt{2+2\cos\theta} - \ell\right)a\overbrace{\frac{\sin\theta}{\sqrt{2+2\cos\theta}}}^{=\sin\frac{1}{2}\theta} = 0}$$

which may be alternatively written as (using goniometry rules):

$$\boxed{4ma^2\ddot{\theta} + (2mga - ka^2)\sin\theta + kla \cdot \sin\tfrac{1}{2}\theta = 0}$$

2.53




$z_1$ = the vertical position of the collar.

The x-y-z- moving frame of reference rotates about the vertical axis.
The rotation $\omega$ of the moving frame of reference is

$$\omega = \dot{\phi}\,\bar{k}$$

The velocity of the centre of mass of the bar, $V_{c,bar}$, then is expressed as

$$\bar{V}_{c,bar} = \bar{V}_{xyz} + \bar{\omega} \times \bar{r}_{rel} + \bar{v}_{rel}$$
$$= \dot{z}_1 \bar{k} + \dot{\phi}\,\bar{k} \times \left( \frac{l}{2} \sin\theta\, \bar{i} - \frac{l}{2} \cos\theta\, \bar{k} \right)$$
$$+ \dot{\theta} \frac{l}{2} \cos\theta\, \bar{i} + \dot{\theta} \frac{l}{2} \sin\theta\, \bar{k}$$

$$\bar{V}_{c,bar} = \dot{z}_1 \bar{k} + \dot{\phi} \frac{l}{2} \sin\theta\, \bar{j}$$
$$+ \dot{\theta} \frac{l}{2} \cos\theta\, \bar{i} + \dot{\theta} \frac{l}{2} \sin\theta\, \bar{k}.$$

$$T = T_m + T_{bar,trans} + T_{bar,rot}$$

$$T_{c,bar,rot} = \frac{1}{2} J_c \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right)$$
$$= \frac{1}{2} \frac{1}{12} M l^2 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right) = \frac{1}{24} M l^2 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta \right)$$

$$T_{c,bar,trans} = \frac{1}{2} M \cdot \bar{V}_{c,bar} \cdot \bar{V}_{c,bar}$$

$$T_m = \frac{1}{2} \cdot m \cdot \dot{z}_1^2$$

$$T_{G,Bar,trans} = \frac{1}{2} M \left( \dot\theta^2 \frac{l^2}{4} \cos^2\theta + \dot\phi^2 \frac{l^2}{4} \sin^2\theta + \left( \dot\theta \frac{l}{2} \sin\theta + \dot Z_1 \right)^2 \right)$$

$$T = \frac{1}{2} m \dot Z_1^2 + \frac{1}{2} M \left[ \dot\theta^2 \frac{l^2}{4} \cos^2\theta + \dot\phi^2 \frac{l^2}{4} \sin^2\theta + \dot\theta^2 \frac{l^2}{4} \sin^2\theta + 2 \dot\theta \frac{l}{2} \sin\theta \dot Z_1 + \dot Z_1^2 \right] + \frac{1}{24} M l^2 \left( \dot\theta^2 + \dot\phi^2 \sin^2\theta \right)$$

$$\Rightarrow T = \frac{1}{2} (m+M) \dot Z_1^2 + \frac{1}{2} M \left[ \frac{\dot\theta^2 l^2}{3} + \dot\phi^2 \frac{l^2}{3} \sin^2\theta + \dot\theta \dot Z_1 l \sin\theta \right]$$

$$V = \frac{1}{2} k_E \cdot Z_1^2 + \frac{1}{2} K_t \phi^2 + m g Z_1 + M g \left( Z_1 - \frac{l}{2} \cos\theta \right)$$

$$L = T - V. \qquad (I)$$

Eqs. of motion :

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot q_k} \right) - \frac{\partial L}{\partial q_k} = 0$$

$$\frac{\partial L}{\partial \dot Z_1} = (m+M) \dot Z_1 + \frac{Ml}{2} \cdot \dot\theta \sin\theta .$$

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot Z_1} \right) = (m+M) \ddot Z_1 + \frac{Ml}{2} \ddot\theta \sin\theta + \frac{Ml}{2} \dot\theta^2 \cos\theta \qquad (1)$$

$$\frac{\partial L}{\partial z} = -K_t z - (m+M) g \quad (2)$$

Combining (1) & (2) with (I), the first equation of motion becomes:

$$\boxed{(m+M)\ddot{z}_1 + \frac{Ml}{2}\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) + K_t z = -(m+M) g \quad (A)}$$

Subsequently:

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{Ml^2\dot{\theta}}{3} + \dot{z}_1 l \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{Ml^2}{3}\ddot{\theta} + \ddot{z} l \sin\theta + \dot{z}_1 l \cos\theta\,\dot{\theta} \quad (3)$$

$$\frac{\partial L}{\partial \theta} = \frac{Ml^2}{3}\dot{\phi}^2\sin\theta\cos\theta + \frac{\dot{\theta}\dot{z}_1 Ml\cos\theta}{2} - \frac{Mgl}{2}\cdot\sin\theta \quad (4)$$

Combining (3) & (4) with (I) gives:

$$\boxed{\frac{Ml^2}{3}\ddot{\theta} + \ddot{z}_1 l\sin\theta + \dot{z}_1\dot{\theta} l\cos\theta - \frac{Ml^2}{3}\dot{\phi}^2\sin\theta\cos\theta - \frac{\dot{\theta}\dot{z}_1 Ml\cos\theta}{2} + \frac{Mgl}{2}\sin\theta = 0 \quad (B)}$$

Finally:

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{Ml^2}{3}\dot{\phi}\sin^2\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{Ml^2}{3}\left(\ddot{\phi}\sin^2\theta + 2\dot{\phi}\sin\theta\cos\theta\,\dot{\theta}\right) \quad (5)$$

4

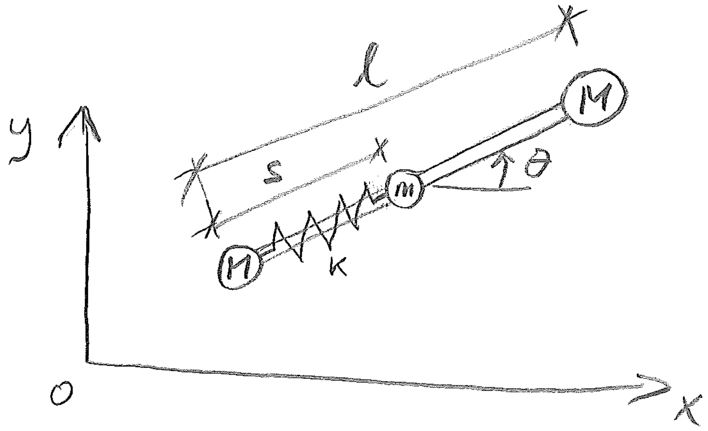
$$\frac{\partial L}{\partial \phi} = -k_t \phi \quad (6)$$

combining (5) & (6) with (I) gives:

$$\boxed{\frac{Ml^2}{3}\left(\ddot{\phi}\sin^2\theta + 2\dot{\phi}\sin\theta\cos\theta\,\dot{\theta}\right) + k_t\phi = 0 \quad (C)}$$

So, the equations of motion of the system are given by (A), (B) & (C).

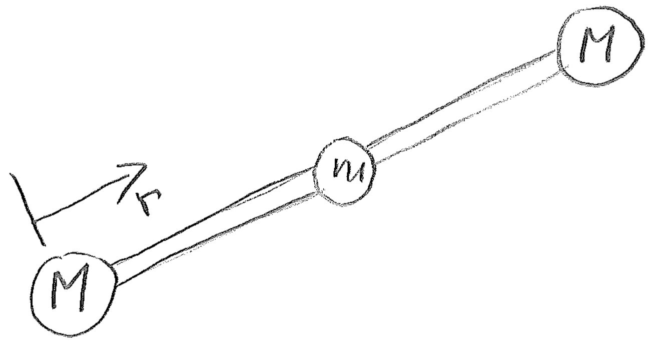
## 2.22




Two masses $M$ with a mass $m$ in between, which is connected to a spring $k$.

The x-y system represent an <u>inertial</u> frame of reference. Compute the potential energy $V$ and the kinetic energy $T$.

The energies are described using the coordinates $x$ and $y$ of the mass centre, together with the generalised coordinates $s$ and $\theta$.




With respect to a coordinate system in the axial direction $r$ of the bar, with its origin at the left mass $M$, the centre of mass $r_c$ can be computed as:

$$r_c = \frac{m \cdot s + M \cdot l + M \cdot 0}{2M + m} = \frac{m \cdot s + M \cdot l}{2M + m}$$

(1)

Subsequently, the relative distance of each mass w.r.t. the centre of mass C is given by (thus assuming a non-inertial frame of reference with its origin at C):

$$\underline{m}: \quad r_{rel,m} = s - \frac{(ms + Ml)}{2M + m}$$

$$= \frac{s(2M+m)}{2M+m} - \frac{ms + Ml}{2M+m}$$

$$= \frac{M(2s - l)}{2M+m} \qquad (2)$$

$$\underline{M_{left}}: \quad r_{rel,M_{left}} = 0 - \frac{(ms + Ml)}{2M+m}$$

$$= \frac{-(ms + Ml)}{2M+m} \qquad (3)$$

$$\underline{M_{right}}: \quad r_{rel,M_{right}} = l - \frac{(ms + Ml)}{2M+m}$$

$$= \frac{l(2M+m)}{2M+m} - \frac{(ms + Ml)}{2M+m}$$

$$= \frac{Ml + m(l-s)}{2M+m} \qquad (4)$$

From (2) - (4), the corresponding relative velocities are calculated as

$$\underline{\underline{m}}: \quad \mathring{r}_{rel,m} = \frac{2M\mathring{s}}{2M+m} \qquad (5)$$

$$\underline{\underline{M_{left}}}: \quad \mathring{r}_{rel,M_{left}} = \frac{-m\mathring{s}}{2M+m} \qquad (6)$$

$$\underline{\underline{M_{right}}}: \quad \mathring{r}_{rel,M_{right}} = \frac{-m\mathring{s}}{2M+m} \qquad (7)$$

The total kinetic energy $T$ is composed of translational and rotational parts as

$$T = T_{trans} + T_{rot} \qquad (7^A)$$

The translational part consists of the kinetic energy associated to the centre of mass, $T^{(1)}_{trans}$, and the kinetic energy of the individual masses about the centre of mass, $T^{(2)}_{trans}$, see Török, pg. 36. Since the position of the centre of mass is decribed by the coordinates $x$ and $y$, the corresponding kinetic energy is:

$$T^{(1)}_{trans} = \frac{1}{2}(2M+m)\cdot(\mathring{x}^2+\mathring{y}^2) \qquad (8)$$

The kinetic energy of the individual masses w.r.t. the centre of mass can be computed using the corresponding relative velocities (5)-(7), i.e.,

$$T^{(2)}_{trans} = \frac{1}{2} m \cdot \frac{4M^2 \dot{s}^2}{(2M+m)^2}$$

$$+ \frac{1}{2} M \cdot \frac{m^2 \dot{s}^2}{(2M+m)^2} * 2$$

$$\Rightarrow T^{(2)}_{trans} = \frac{2mM^2 \dot{s}^2 + Mm^2 \dot{s}^2}{(2M+m)^2}$$

$$= \frac{mM \cdot \dot{s}^2}{2M+m} \qquad (9)$$

From (8) and (9), the total translational kinetic energy becomes

$$T_{trans} = \frac{2M+m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{mM \cdot \dot{s}^2}{2M+m} \qquad (10)$$

The rotational kinetic energy follows from

$$T_{rot} = \frac{1}{2} I_c \cdot \dot{\theta}^2 \qquad (11)$$

where $I_c$ is the mass moment of inertia about the centre of mass.

Using the relative distances (2)-(4) of the individual masses w.r.t. the centre of mass C, the mass moment of inertia $I_c$ can be computed as:

$$I_c = m \cdot \left[\frac{M(2s-l)}{2M+m}\right]^2 + M \cdot \left[\frac{-(ms+Ml)}{2M+m}\right]^2 + M \cdot \left[\frac{Ml+m(l-s)}{2M+m}\right]^2 \qquad (12)$$

This expression can be further developed as

$$I_c = \frac{M}{(2M+m)^2}\Big\{ mM(2s-l)^2 + m^2s^2 + 2mMsl + M^2l^2 + M^2l^2 + 2Mml(l-s) + m^2(l-s)^2 \Big\} \qquad (13)$$

Writing the first term between curly braces as

$$mM(2s-l)^2 = 2Mm(l-s)^2 - Mml^2 + 2Mms^2. \qquad (14)$$

and inserting (14) into (13) gives

$$I_c = \frac{M}{(2M+m)^2} \left\{ (2M+m)\left(Ml^2 + m(l-s)^2 + ms^2\right) \right\}$$

$$= \frac{M}{2M+m} \left( Ml^2 + m(l-s)^2 + ms^2 \right) \quad (15)$$

Substituting (15) into (11) gives

$$T_{rot} = \frac{M}{2(2M+m)} \left( Ml^2 + m(l-s)^2 + ms^2 \right) \dot{\theta}^2 \quad (16)$$

Finally, inserting the rotational kinetic energy (16) and the translational kinetic energy (10) into (7A), the total kinetic energy becomes

$$T = \frac{2M+m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{mM\,\dot{s}^2}{2M+m}$$

$$+ \frac{M}{2(2M+m)} \left( Ml^2 + m(l-s)^2 + ms^2 \right) \dot{\theta}^2 \quad (17)$$

The total potential energy is straightforwardly computed from the spring deformation, $\frac{l}{2} - s$, as

$$V = \frac{1}{2} k \left( \frac{l}{2} - s \right)^2 \quad (18).$$