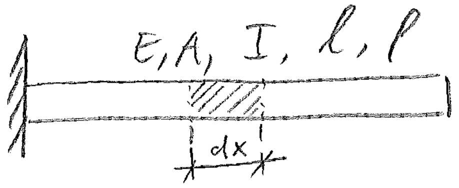
Example problem ① week 7



Equations of motion? Boundary conditions for $x = 0$ and $x = l$?

The vertical displacement of the beam is given by $w = \hat{w}(x,t)$, where $w(0,t) = 0$ and $w_x(0,t) = 0$.

* Incremental mass: $dm = \rho \cdot A\,dx$
* Incremental kinetic energy: $dT = \frac{1}{2} \cdot \dot{w}^2\, dm$
$$= \frac{1}{2} \cdot w_t^2\, \rho A\, dx$$
* Incremental potential energy: $dV = \frac{1}{2} E I \kappa^2\, dx$
$$= \frac{1}{2} EI \cdot w_{xx}^2\, dx$$
where $w_{xx} = \dfrac{\partial^2 w}{\partial x^2}$

$$L = \int_0^l dL = \int_0^l dT - dV$$
$$= \int_0^l \left(\frac{1}{2}\rho A w_t^2 - \frac{1}{2} EI w_{xx}^2\right) dx$$

The "action" now becomes

$$I(w) = \int_{t_0}^{t_1} L\, dt = \int_{t_0}^{t_1}\int_0^l \underbrace{\left(\frac{1}{2}\rho A w_t^2 - \frac{1}{2} EI w_{xx}^2\right)}_{=\text{ Lagrangian density } \mathcal{L}.} dx$$

The function w for which the action $I(w)$ becomes minimal is found through requiring:

$$\delta I(w) = 0$$

Using the Expressions derived during a lecture for the case

$$I = \int_{t_0}^{t_1} \int_{x_0}^{x_1} F(x, t, u, u_t, u_{xx})\, dx\, dt,$$

which, for $\delta I = 0$, are

$$\delta I = \int_{t_0}^{t_1} \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial u_t}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right)\right) \delta u \cdot dx\, dt$$

$$+ \int_{x_0}^{x_1} \frac{\partial F}{\partial u_t} \delta u \Big|_{t_0}^{t_1} dx + \int_{t_0}^{t_1} \frac{\partial F}{\partial u_{xx}} \delta u_x \Big|_{x_0}^{x_1} dt$$

$$- \int_{t_0}^{t_1} \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{xx}}\right) \delta u \Big|_{x_0}^{x_1} dt = 0$$

Interpreting F as the Lagrangian density $\mathcal{L}$ and u as the vertical displacement w then gives

$$\delta I = \int_{t_0}^{t_1}\int_0^l \left(-\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial w_t}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right)\right) \delta w \, dx\, dt \quad (1)$$

$$+ \int_0^l \frac{\partial \mathcal{L}}{\partial w_t} \delta w \Big|_{t_0}^{t_1} dx \quad (2)$$

$$+ \int_{t_0}^{t_1} \frac{\partial \mathcal{L}}{\partial w_{xx}} \delta w_x \Big|_0^l dt \quad (3)$$

$$- \int_{t_0}^{t_1} \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) \delta w \Big|_0^l dt = 0 \quad (4)$$

So, with the Fundamental Lemma on pg. 166 of Török, for (1) $=0$, we obtain

$$-\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial w_t}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right) = 0$$

with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\rho A\, w_t^2 - \tfrac{1}{2} EI\, w_{xx}^2$$

this leads to

$$-\rho A\, w_{tt} - EI\, w_{xxxx} = 0$$

$$\Rightarrow \boxed{\rho A \cdot \frac{\partial^2 w}{\partial t^2} + EI \cdot \frac{\partial^4 w}{\partial x^4} = 0} \qquad (\mathrm{I})$$

which is the equation of motion, or the wave equation, of the beam.

In addition:

$$② = 0 : \quad \frac{\partial \mathcal{L}}{\partial w_t}\, \delta w \Big|_{t_0}^{t_1} = 0 \qquad (\mathrm{II})$$

Hence $\delta w(x, t_0) = \delta w(x, t_1) = 0$
from the initial and final conditions prescribed at $t = t_0$ & $t = t_1$, respectively.
Alternatively, $\frac{\partial \mathcal{L}}{\partial w_t} = 0 \rightarrow w_t = 0$ at $t = t_1$ and t_2.

$$③ = 0 : \quad \frac{\partial \mathcal{L}}{\partial w_{xx}}\, \delta w_x \Big|_0^l = 0 \qquad (\mathrm{III})$$

Since $w_x(0,t) = 0 \Rightarrow \delta w_x(0,t) = 0$.

In addition, since $\delta w_x(l,t) \neq 0$,

we need to have

$$\frac{\partial \mathcal{L}}{\partial w_{xx}}\Big|_{x=l} = 0$$

$$\rightarrow -EI\, w_{xx}\Big|_{x=l} = 0$$

(which can be interpreted as the bending moment at $x = l$ being zero.)

4 $= 0$: $$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right)\delta w\Big|_0^l = 0 \qquad (\mathrm{IV})$$

Since $w(0,t) = 0 \Rightarrow \delta w(0,t) = 0$

Also, since $\delta w(l,t) \neq 0$, we need to have that $$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial w_{xx}}\right)\Big|_{x=l} = 0$$

$$\Rightarrow \quad -EI\, w_{xxx}\Big|_{x=l} = 0$$

(which can be interpreted as the shear force at $x = l$ being zero).

So, the above framework provides the equation of motion (I), together with the initial conditions (II), and the boundary conditions (III) and (IV). In other words, the complete formulation of the problem is found with the above method.