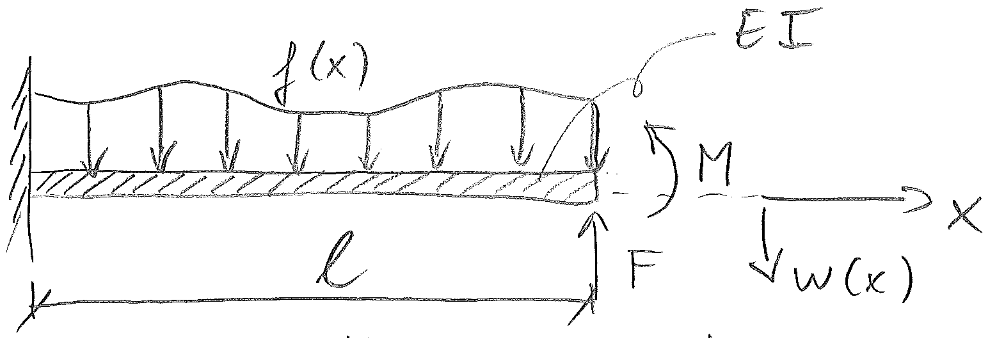
Example problem 2, week 7



Equilibrium equation and boundary conditions?

The problem is static, so the vertical displacement $w = \hat{w}(x)$.

For static problems Hamiltons principle

$$\int_{t_0}^{t_1} \delta L \, dt = \int_{t_0}^{t_1} (\delta T - \delta V) \, dt = 0$$

turns into

$$\delta V = 0 \qquad (= \text{stationary potential energy})$$

Since the kinetic energy $T = 0$ (and thus $\delta T = 0$) and time t is irrelevant, so that the time integral vanishes.

The virtual work generated by the external loads is

$$\delta W = \int_0^l f(x)\, \delta w(x)\, dx - F \delta w(l) - M \delta w_x(l).$$

For conservative systems we have

$$\delta W = -\delta V^{force}$$

So that

$$\delta V^{force} = \int_0^{\ell} -f(x)\,\delta w(x)\,dx + F\delta w(\ell) + M\delta w_x(\ell).$$

from which the potential energy due to the applied forces becomes:

$$V^{force} = \int_0^{\ell} -f(x)\,w(x)\,dx + Fw(\ell) + Mw_x(\ell)$$

In addition, the strain energy of the beam is:

$$V^{strain} = \int_0^{\ell} \frac{1}{2} EI\,\kappa^2\,dx$$

$$= \int_0^{\ell} \frac{1}{2} EI\,w_{xx}^2\,dx$$

The total potential energy V then becomes

$$V = V^{strain} + V^{force}$$

$$= \int_0^{\ell}\left(\frac{1}{2} EI\,w_{xx}^2 - f(x)\,w(x)\right)dx + Fw(\ell) + Mw_x(\ell).$$

Now, with this expression, $\delta V = 0$ turns into:

$$\delta V = \int_0^l \left(EI\, w_{xx}\, \delta w_{xx} - f(x)\, \delta w \right) dx$$

$$+ F\, \delta w(l) + M\, \delta w_x(l) = 0$$

With integration by parts, we obtain

$$\Rightarrow \delta V = EI\, w_{xx}\, \delta w_x \Big|_0^l - \int_0^l EI\, w_{xxx}\, \delta w_x\, dx$$

$$- \int_0^l f(x)\, \delta w\, dx + F\, \delta w(l) + M\, \delta w_x(l) = 0$$

Since $w(0) = 0$ and $w_x(0) = 0$, and therefore $\delta w(0) = 0$ and $\delta w_x(0) = 0$, the above expression may be written as (when using again integration by parts for the 2nd term):

$$\delta V = \left(EI\, w_{xx} + M \right) \delta w_x \Big|_{x=l} \quad (1)$$

$$- \left(EI\, w_{xxx} - F \right) \delta w \Big|_{x=l} \quad (2)$$

$$+ \underbrace{\int_0^l \left(EI\, w_{xxxx} - f(x) \right) \delta w}_{(3)} = 0$$

$① = 0$: $\left(M = - EI\, w_{xx} \right) \Big|_{x=l}$ (I)

$② = 0$: $\left(F = EI\, w_{xxx} \right) \Big|_{x=l}$ (II)

$③ = 0$: $EI\, w_{xxxx} = f(x)$ (III) for $0 < x < l$

So, with the (natural) boundary conditions (I) and (II) at $x = l$, and the (essential) boundary conditions $w = 0$; $w_x = 0$ at $x = 0$, and the governing equation (III) for the whole domain, the complete problem is determined. .