Recursive BLUE

derivation for AE4-E18 Principles of Navigation by C. Tiberius and R. van Bree, March 2009

1 one epoch

m_1 -vector of observables $\underline{y}_1$, n-vector of unknown parameters x.

$$E(\underline{y}_1) = A_1 x \qquad \text{with observables' variance matrix } \; Q_{y_1} \tag{1}$$

BLUE of parameter vector x reads

$$\underline{\hat{x}}_1 = \left(A_1^T Q_{y_1}^{-1} A_1\right)^{-1} A_1^T Q_{y_1}^{-1} \underline{y}_1 \tag{2}$$

with variance matrix

$$Q_{\hat{x}_1} = \left(A_1^T Q_{y_1}^{-1} A_1\right)^{-1} \tag{3}$$

The subscript 1 with $\underline{\hat{x}}$, hence $\underline{\hat{x}}_1$, denotes the fact that this estimator is based on $\underline{y}_1$.
Later we will need eq. (2) as

$$\left(A_1^T Q_{y_1}^{-1} A_1\right) \underline{\hat{x}}_1 = A_1^T Q_{y_1}^{-1} \underline{y}_1 \tag{4}$$

2 two epochs

m_1-vector of observables $\underline{y}_1$ and m_2-vector of observables $\underline{y}_2$ are both related to the (same) vector x

$$E\begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} x \qquad \text{with variance matrix} \qquad \begin{pmatrix} Q_{y_1} & 0 \\ 0 & Q_{y_2} \end{pmatrix} \tag{5}$$

BLUE of parameter x reads

$$\begin{aligned} \underline{\hat{x}}_2 &= \left(\begin{pmatrix} A_1^T & A_2^T \end{pmatrix} \begin{pmatrix} Q_{y_1}^{-1} & \\ & Q_{y_2}^{-1} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \right)^{-1} \begin{pmatrix} A_1^T & A_2^T \end{pmatrix} \begin{pmatrix} Q_{y_1}^{-1} & \\ & Q_{y_2}^{-1} \end{pmatrix} \begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \end{pmatrix} \\ &= \left(A_1^T Q_{y_1}^{-1} A_1 + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} \left(A_1^T Q_{y_1}^{-1} \underline{y}_1 + A_2^T Q_{y_2}^{-1} \underline{y}_2\right) \end{aligned} \tag{6}$$

with variance matrix

$$Q_{\hat{x}_2} = \left(\begin{pmatrix} A_1^T & A_2^T \end{pmatrix} \begin{pmatrix} Q_{y_1}^{-1} & \\ & Q_{y_2}^{-1} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \right)^{-1} = \left(A_1^T Q_{y_1}^{-1} A_1 + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} \tag{7}$$

The subscript 2 with $\underline{\hat{x}}$ denotes the fact that this estimator is based on all vectors of observables up to, and including $\underline{y}_2$, thus in this case $\underline{y}_1$ and $\underline{y}_2$.

In the next two sections we will develop (6) and (7) such that they are expressed in terms of the 'old' state $\underline{\hat{x}}_1$ (rather than the original observables $\underline{y}_1$) and the new observables $\underline{y}_2$.

3 recursion: information form

Substitute the inverse of (3) in the first term of (6), and relation (4) in the second term of (6), again with the inverse of (3)

$$\underline{\hat{x}}_2 = \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} \left(Q_{\hat{x}_1}^{-1} \underline{\hat{x}}_1 + A_2^T Q_{y_2}^{-1} \underline{y}_2\right) \tag{8}$$

next, add and subtract $A_2^T Q_{y_2}^{-1} A_2 \underline{\hat{x}}_1$ to the second term

$$\begin{aligned} \underline{\hat{x}}_2 &= \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} \left(Q_{\hat{x}_1}^{-1} \underline{\hat{x}}_1 + A_2^T Q_{y_2}^{-1} A_2 \underline{\hat{x}}_1 + A_2^T Q_{y_2}^{-1} \underline{y}_2 - A_2^T Q_{y_2}^{-1} A_2 \underline{\hat{x}}_1\right) \\ \underline{\hat{x}}_2 &= \underline{\hat{x}}_1 + \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} A_2^T Q_{y_2}^{-1} (\underline{y}_2 - A_2 \underline{\hat{x}}_1) \qquad (9) \end{aligned}$$

The variance matrix follows from (7) with the inverse of (3)

$$Q_{\hat{x}_2} = \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} \tag{10}$$

Eqns (9) and (10) are known as the information form of recursive BLUE.

4 intermezzo-1

The following matrix identity

$$(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T)(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T)^{-1} = I_{m_2}$$

is pre-multiplied by $A_2^T Q_{y_2}^{-1}$ (on both sides), and next the term $Q_{\hat{x}_1}^{-1} Q_{\hat{x}_1}$, being the identity matrix, is inserted right in front of the first matrix within the first pair of brackets

$$\left(Q_{\hat{x}_1}^{-1} Q_{\hat{x}_1} A_2^T Q_{y_2}^{-1} Q_{y_2} + A_2^T Q_{y_2}^{-1} A_2 Q_{\hat{x}_1} A_2^T\right) \left(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T\right)^{-1} = A_2^T Q_{y_2}^{-1}$$

then move $Q_{\hat{x}_1} A_2^T$ outside (to the back) of the first pair of brackets

$$\left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right) Q_{\hat{x}_1} A_2^T \left(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T\right)^{-1} = A_2^T Q_{y_2}^{-1}$$

and the matrix of the first pair of brackets is brought to the right-hand side

$$Q_{\hat{x}_1} A_2^T \left(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T\right)^{-1} = \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} A_2^T Q_{y_2}^{-1} \tag{11}$$

5 intermezzo-2

Post-multiply (both sides of) eq. (11) by $-A_2 Q_{\hat{x}_1}$, and next add $Q_{\hat{x}_1}$ to both sides

$$Q_{\hat{x}_1} - Q_{\hat{x}_1} A_2^T \left(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T\right)^{-1} A_2 Q_{\hat{x}_1} = -\left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} A_2^T Q_{y_2}^{-1} A_2 Q_{\hat{x}_1} + Q_{\hat{x}_1}$$

re-arrange the right-hand side, and insert a matrix identity just in front of the very last $Q_{\hat{x}_1}$

$$= \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} \left(-A_2^T Q_{y_2}^{-1} A_2\right) Q_{\hat{x}_1} + \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right) Q_{\hat{x}_1}$$

collecting terms together on the right hand side, and using $Q_{\hat{x}_1}^{-1} Q_{\hat{x}_1} = I_n$

$$Q_{\hat{x}_1} - Q_{\hat{x}_1} A_2^T \left(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T\right)^{-1} A_2 Q_{\hat{x}_1} = \left(Q_{\hat{x}_1}^{-1} + A_2^T Q_{y_2}^{-1} A_2\right)^{-1} \tag{12}$$

6 recursion: variance form

Using (11) in (9) yields

$$\underline{\hat{x}}_2 = \underline{\hat{x}}_1 + Q_{\hat{x}_1} A_2^T \left(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T\right)^{-1} \left(\underline{y}_2 - A_2 \underline{\hat{x}}_1\right) \tag{13}$$

Using (12) in (10) yields

$$Q_{\hat{x}_2} = Q_{\hat{x}_1} - Q_{\hat{x}_1} A_2^T \left(Q_{y_2} + A_2 Q_{\hat{x}_1} A_2^T\right)^{-1} A_2 Q_{\hat{x}_1} \tag{14}$$

Eqns. (13) and (14) are known as the variance form of recursive BLUE.