Kinematics of a spinning body when the direction of the spin axis changes

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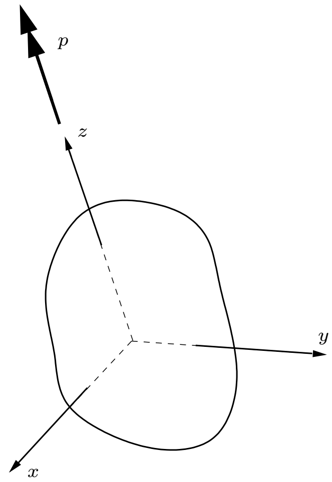
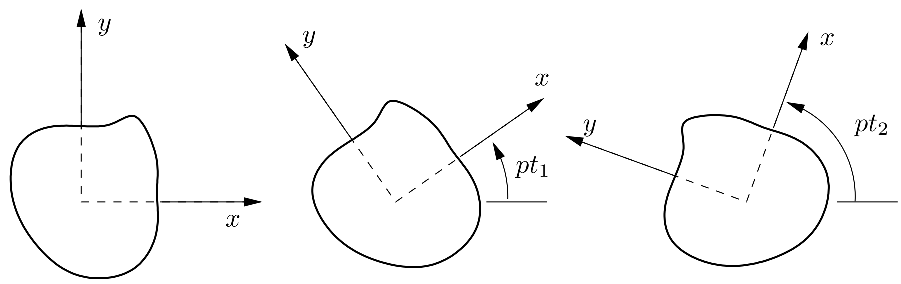


Consider a body with an attached xyz-coordinate system. The coordinate system is such that the axes coincide with principal directions and the origin is either a fixed point or the mass centre. The body is set to spin about the z-axis with constant rate p. It is immediate to realise that the components of the angular velocity vector $\boldsymbol{\omega}$ are

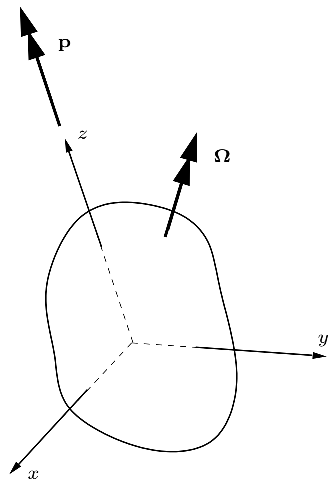
$$\omega_1 = \omega_2 = 0; \qquad \omega_3 = p. \tag{1}$$

If no further action is exerted on the body, one would observe that the z-axis would keep the current orientation and the xy-plane (which is indeed attached to the body) would rotate with rate p about the z-axis. An external observer standing on top of the z-axis would make the following snapshots for $t = 0$, $t = t_1$ and $t = t_2$:



The components of $\boldsymbol{\omega}$ do not change during this motion. Consequently, it can be stated that

$$\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0. \tag{2}$$



An action is now exerted on the body in order to change the direction of the spin axis —the z-axis in this case— with constant rate. This action is often referred to as *precession* and is described by an angular velocity vector $\boldsymbol{\Omega} = \Omega_1\mathbf{i} + \Omega_2\mathbf{j} + \Omega_3\mathbf{k}$ superposed to the initial spin $\mathbf{p} = p\mathbf{k}$. The major difference between $\mathbf{\Omega}$ and $\mathbf{p}$ is that while $\mathbf{p}$ is attached to the body, $\mathbf{\Omega}$ is fixed. The latter observation is formalised as

$$\frac{d\mathbf{\Omega}}{dt} = \mathbf{0}, \tag{3}$$

while the angular velocity $\boldsymbol{\omega}$ of the coordinate system attached to the body now has the components

$$\omega_1 = \Omega_1; \qquad \omega_2 = \Omega_2; \qquad \omega_3 = \Omega_3 + p. \tag{4}$$

It can therefore be expected that since the coordinate system is rotating while the fraction $\mathbf{\Omega}$ of the angular velocity is a fixed vector in space, the components of $\mathbf{\Omega}$ will change in time. Equation (3) can now be developed as

$$\frac{d\mathbf{\Omega}}{dt} = \dot{\Omega}_1\mathbf{i} + \dot{\Omega}_2\mathbf{j} + \dot{\Omega}_3\mathbf{k} + \boldsymbol{\omega} \times \mathbf{\Omega} = \mathbf{0}. \tag{5}$$

The cross product in (5) is elaborated as

$$\boldsymbol{\omega} \times \mathbf{\Omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Omega_1 & \Omega_2 & \Omega_3 + p \\ \Omega_1 & \Omega_2 & \Omega_3 \end{vmatrix} = -p\Omega_2\mathbf{i} + p\Omega_1\mathbf{j}, \tag{6}$$

which, after back-substitution in (5), provides the expressions

$$\dot{\Omega}_1 = p\Omega_2; \qquad \dot{\Omega}_2 = -p\Omega_1; \qquad \dot{\Omega}_3 = 0. \tag{7}$$

Remember that the total angular velocity vector $\boldsymbol{\omega}$ of the body has the components

$$\omega_1 = \Omega_1; \qquad \omega_2 = \Omega_2; \qquad \omega_3 = \Omega_3 + p. \tag{8}$$

Taking time derivatives in (8) one gets

$$\dot{\omega}_1 = \dot{\Omega}_1; \qquad \dot{\omega}_2 = \dot{\Omega}_2; \qquad \dot{\omega}_3 = \dot{\Omega}_3 + \dot{p}. \tag{9}$$

Substituting (7) into (9) and keeping in mind that the spin rate is constant, i.e. $\dot{p} = 0$, provides

$$\dot{\omega}_1 = p\Omega_2; \qquad \dot{\omega}_2 = -p\Omega_1; \qquad \dot{\omega}_3 = 0. \tag{10}$$

Finally, equation (8) can be used in (10) to provide a purely kinematic relation between the angular velocity and acceleration components,

$$\dot{\omega}_1 = p\omega_2; \qquad \dot{\omega}_2 = -p\omega_1; \qquad \dot{\omega}_3 = 0. \tag{11}$$

In practice one should identify the spin vector $\mathbf{p}$ and the action $\mathbf{\Omega}$ at the considered instant, choose convenient axes —the z-axis parallel to $\mathbf{p}$ and the x- and y-axes in such a way that the components ω_1 and ω_2 have a simple expression— and obtain the angular accelerations $\dot{\omega}_1$ and $\dot{\omega}_2$ from expression (11). All components $\{\omega_i\}$ and $\{\dot{\omega}_i\}$ are then available and can be plugged into Euler equations of motion. The calculated moments are then expressed in the chosen axes as well.