Ginzburg-Landau equations

- Order parameter
- Free energy
- Ginzburg-Landau equations
- Critical field of a film
- Type-II superconductors
- Upper critical field
- Lower critical field
- Vortex structure
- Vortex motion. Pinning

Ginzburg-Landau theory

BCS: microscopic, very powerful – but too complicated

London equations: describe a very special situation $\xi \ll \delta$

Way out: to develop a theory based on Landau theory of second order phase transitions.

Applicability range: close to the transition temperature.
No restrictions concerning the nature of (conventional) superconductors.

Idea: order parameter is a complex number – wave function of Cooper pairs.

$$\Psi(\vec{r})$$

Free energy

Free energy – functional of the order parameter. $\tau = (T - T_c)/T_c$

$$F_s = F_n + \int dV \left\{ a\tau|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{1}{4m}\left|\left(-i\hbar\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\Psi\right|^2 + \frac{H^2}{8\pi} \right\}$$

Energy of the normal state

Expansion at the critical point

Fluctuation contribution ("kinetic energy of Cooper pairs")

Energy of magnetic field

Without fluctuations: $|\Psi_0|^2 = \begin{cases} -a\tau/b & T < T_c \\ 0 & T > T_c \end{cases}$

Ginzburg-Landau equations

Minimizing the free energy:

$$F_s = F_n + \int dV \left\{ a\tau|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{1}{4m}\left|\left(-i\hbar\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\Psi\right|^2 + \frac{H^2}{8\pi} \right\}$$

With respect to Ψ^*

$$a\tau\Psi + b|\Psi|^2\Psi + \frac{1}{4m}\left(-i\hbar\vec{\nabla} - \frac{2e}{c}\vec{A}\right)^2\Psi = 0$$

Boundary condition: $\vec{n}\left(-i\hbar\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\Psi = 0$

Ginzburg-Landau equations

Minimizing the free energy:

$$F_s = F_n + \int dV \left\{ a\tau|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{1}{4m}\left|\left(-i\hbar\vec{\nabla} - \frac{2e}{c}\vec{A}\right)\Psi\right|^2 + \frac{H^2}{8\pi} \right\}$$

With respect to $\vec{A}$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{j} \qquad \vec{j} = -\frac{ie\hbar}{2m}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) - \frac{2e^2}{mc}|\Psi|^2\vec{A}$$

"Maxwell equation"

Rescaling GL equations

$$\Psi \to \Psi/\Psi_0, r \to r/\delta, H \to H/H_0, H_0 = 2\sqrt{2\pi}a\tau b^{3/2}$$

The rescaled equations contain only one constant! $\kappa = 2eH_0\delta^2/\hbar c$

$$(-i\kappa^{-1}\vec{\nabla} - \vec{A})^2\Psi - \Psi + \Psi|\Psi|^2 = 0$$

$$\vec{n}\left(-i\kappa^{-1}\vec{\nabla} - \vec{A}\right)\Psi = 0 \text{ at the surface}$$

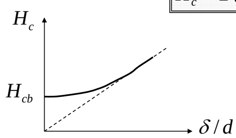
$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = -\frac{i}{2\kappa}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) - |\Psi|^2\vec{A}$$

Critical field of a film

Steps of the calculation:
- Write the energy difference between S and N phases;
- Find at which field the energies are the same (depends on Ψ)
- Solve GL equations and find the order parameter;
- Find the critical field

Results: Thick films, $d \gg \delta$ $\boxed{H_c = H_{cb}(1+\delta/d)}$ $\boxed{H_{cb} = H_0/\sqrt{2}}$

Thin films $d \ll \delta$ $\boxed{H_c = 2\sqrt{6}H_{cb}\delta/d}$



Critical field of a film

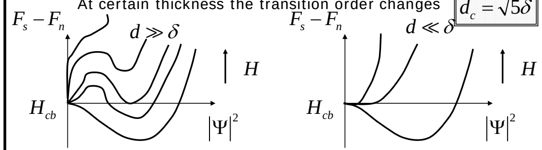
Let us now look at the details.

GL equations for $\kappa \ll 1$ give $\Psi \approx \text{const}$

Thick films $d \gg \delta$ $\boxed{|\Psi|^2 \approx 1}$ First-order phase transition

Thin films $d \ll \delta$ $\boxed{\Psi = 0}$ Second-order phase transition

At certain thickness the transition order changes $\boxed{d_c = \sqrt{5}\delta}$



Critical field of a film

Type-I superconductors: 1st order phase transition in the bulk is realized by splitting into N and S layers

Interface energy: positive
Critical field of one layer is higher

Optimal thickness of a layer

Intermediate state!

Type II superconductors: does not work!

Interface energy: negative

Formation of infinitely thin layers is preferable

First-order phase transition is shifted to infinitely strong fields!

Type-II superconductors

Phase transition must be of the second order and occurs in the interval of fields

$H > H_{c2}$ (Upper critical field)
Even a small piece of superconductor is unstable
Normal metal

$H < H_{c1}$ (Lower critical field)
Fully developed Meissner effect
Superconductor

$H_{c1} < H < H_{c2}$ Mixed state
No Meissner effect

Upper critical field

Two possible mechanisms:
- Spins of two electrons in a Cooper pair become parallel – too strong fields
- Larmor radius of a Cooper pair becomes shorter than ξ

Larmor radius: $r_L = \dfrac{cp_\perp}{eH} < \dfrac{\hbar c}{eH\xi}$

$p_\perp$ -component of momentum of the pair transverse to the magnetic field

$\boxed{\xi < r_L \Rightarrow H < H_{c2} \sim \dfrac{\hbar c}{e\xi^2}}$ $\boxed{H_{c2} = \kappa H_{cb} > H_{cb}}$

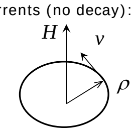
Lower critical field

What happens below H_{c2}?

Magnetic field penetrates the superconductor

Persistent currents (no decay): vortices!

Quantum vortices:



Bohr-Sommerfeld quantization:

$$2\pi n\hbar = \oint p\,dq = 2m\oint \vec{v}\,d\vec{l} = 2mv \cdot 2\pi\rho$$

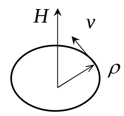
$\boxed{v = \hbar n/2m\rho}$ - quantized velocity

n=1: energetically the most favorable $\boxed{\xi \ll \rho \ll \delta}$

Lower critical field

Vortex energy (per unit length): $v = \hbar n / 2m\rho$ $\xi \ll \rho \ll \delta$

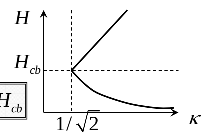
$$E = \frac{n_s}{2}\int_{\xi}^{\delta} 2mv_s^2 \cdot 2\pi\rho d\rho = \frac{\pi n_s \hbar^2}{4m}\ln\frac{\delta}{\xi} = \frac{\pi n_s \hbar^2}{4m}\ln\kappa$$



Vortex magnerization (per unit length):

$$M = \frac{n_s}{2}\frac{2e}{c}\int_{\xi}^{\delta}\rho v_s \cdot 2\pi\rho d\rho = \frac{\pi n_s e\hbar}{2mc}\delta^2$$

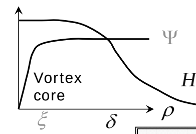
Magnetic energy: *-MH*



Lower critical field: the first vortex is formed

$$H_{c1} = E/M = (c\hbar / e\delta^2)\ln\kappa \sim H_{cb}\ln\kappa/\kappa^2 < H_{cb}$$

Isolated vortex



What is the magnetic flux in a vortex?

Current:

$$\vec{j} = -\frac{ie\hbar}{2m}\left(\Psi^*\vec{\nabla}\Psi - \Psi\vec{\nabla}\Psi^*\right) - \frac{2e^2}{mc}|\Psi|^2\vec{A}$$

$\Psi = \Psi e^{i\theta}$ ⇒ $\vec{j} = |\Psi|^2\left(\frac{e\hbar}{m}\vec{\nabla}\theta - \frac{2e^2}{mc}\vec{A}\right)$

Far from the vortex core: no current

Let us integrate over a contour around the vortex center.

$$\hbar\oint\vec{\nabla}\theta d\vec{l} = \frac{2e}{c}\oint\vec{A}d\vec{l} = \frac{2e\Phi}{c}$$

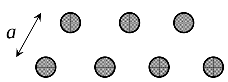
$\Phi = n\Phi_s, \Phi_s \equiv \pi\hbar c/e$ -superconducting flux quantum

Vortex lattice

Close to lower critical field – sparse vortices

Form a triangular vortex lattice

Lattice period: determined by ***H***



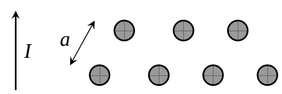
Per plaquette: half a vortex

Total flux per plaquette: $\frac{\Phi_s}{2} = \frac{\sqrt{3}}{4}a^2 H$

H_{c2} : vortices overlap $a \sim \xi \Rightarrow H_{c2} \sim \Phi_s/\xi^2$

$$H_{c1} \sim \Phi_s/\delta^2; H_{cb} \sim \Phi_s/\delta\xi$$

Vortex motion



If current is passed through a superconductor in a mixed state: Lorentz force acts at the core electrons

$\vec{F} \propto \vec{j} \times \vec{H}$ Vortex motion transverse to the current

Motion of magnetic flux with a constant velocity

Electric field $\vec{E} = \frac{1}{c}\vec{H} \times \vec{v}$

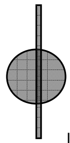
Resistance!! $\rho \sim \rho_N H / H_{c2}$ Like if produced by vortex cores.

Pinning

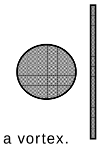
Does it mean Type-II superconductors have resistance down to very low fields?

No, vortices do not move (at weak currents)

Pinning:



Is more favorable than



It costs energy to depin a vortex.