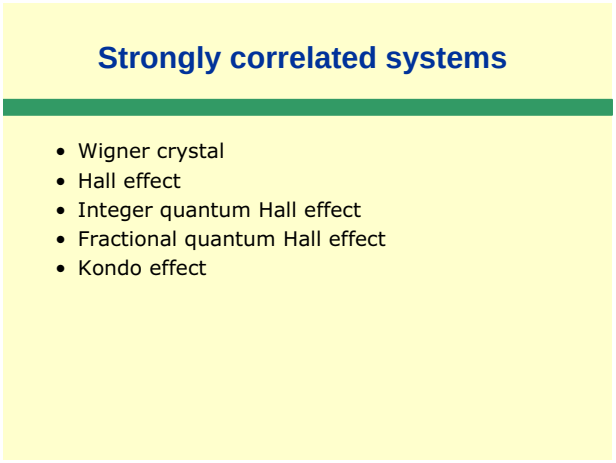
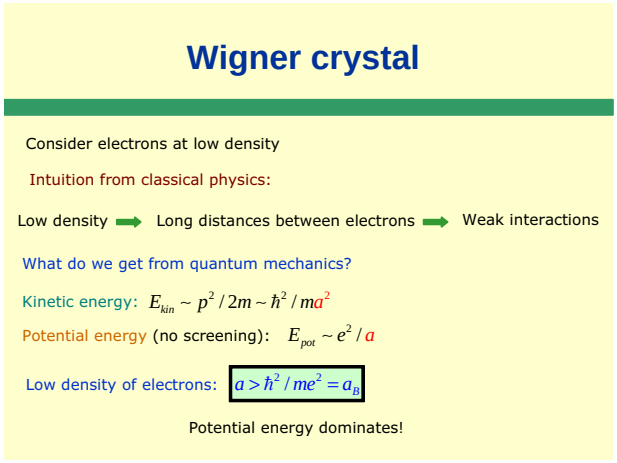
I. STRONGLY CORRELATED SYSTEMS

This lecture is devoted to strongly correlated systems — those where interactions between electrons are important. So far, we only encountered interactions on two occasions: in Lecture 2 to evaluate the typical scattering time, and in Lecture 8 to explain the superconductivity. These two occasions were qualitatively different: Electron-electron interactions in metals are weak (actually, they are strong, but then the Fermi liquid theory takes care of it and explains that quasiparticles in Fermi liquid behave almost like free electrons), and we could treat interactions in perturbation theory. In contrast, BCS theory of superconductivity does not treat interactions perturbatively, and as a result we were able to establish that a new ground state — the superconducting state — of the system appears, and the properties of this new state are quite different from a normal metal. This is a common situation for a strongly interacting system — interactions between electrons lead to a formation of a new state of matter.

Below, we consider three examples of strongly interacting systems — Wigner crystal, fractional quantum Hall effect, and Kondo effect. For strongly correlated systems, there are no general methods of solution. Sometimes, the systems can be solved exactly. Sometimes, one can get an information from qualitative considerations. Sometimes, numerical studies are the only way. This is why this Lecture is more qualitative than the others: we will consider systems which are really complex, but in most cases will only employ qualitative arguments and simple estimates.





A. Wigner crystal

Consider an electron system at low density. If electrons were classical particles, interactions at low density would be negligible. Indeed, low density means long distances between electrons, and since the interactions decay with the distance (as inverse distance for Coulomb law, and exponentially if screening is effective), one could disregard interactions or treat them as perturbation.

But electrons are quantum particles, and this makes the story different. Indeed, let a be the average distance between the electrons (it is related to the density n as $a \sim n^{-1/3}$). For Coulomb interactions, the typical scale of the potential energy is $E_{pot} \sim e^2/a$. The kinetic energy is $p^2/2m$, where p is the particle momentum. However, electrons are Fermi-particles, and the relevant values of the momenta lie close to the Fermi surface. The value of the Fermi momentum p_F depends on the electron density and can be estimated from the Heisenberg uncertainty relation as $p_F \sim \hbar/a$. Thus, the kinetic energy is of the order of $E_{kin} \sim \hbar^2/(2ma^2)$. For low densities (large distances a) the potential energy is small, as expected, but the kinetic energy is even smaller!! It decays as a^{-2} and at very long distances a becomes insignificant. Electrons at low densities represent a strongly correlated system.

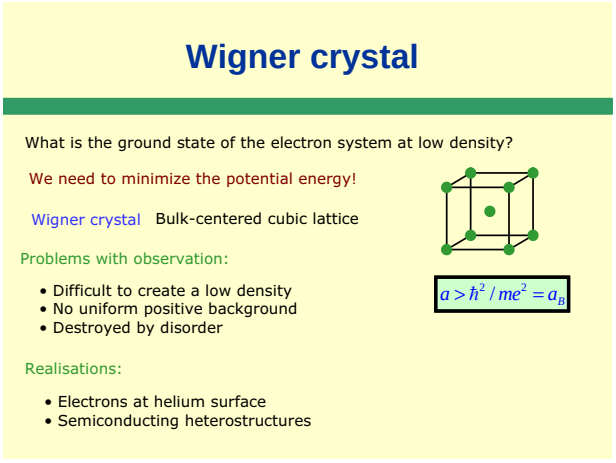
What is then the ground state of such system? If the kinetic energy is insignificant, one has to minimize the potential energy. It is natural to assume (and this is confirmed by numerical calculations) that the electron configuration minimizing the potential energy is a *crystalline lattice*. In particular, in three dimensions it is bulk-centered cubic lattice. This phase is known as *Wigner crystal* and has nothing similar to the properties of free electron gas, which we considered in Lecture 1.

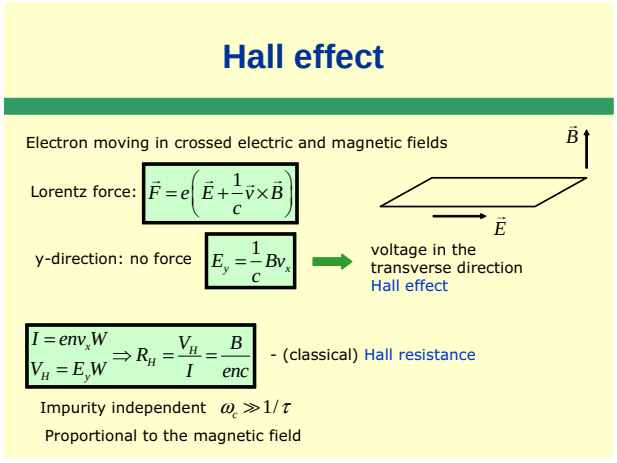
Wigner crystal is stable at extremely low densities. If the density increases, the kinetic energy becomes important, and eventually the crystal melts.

Wigner crystal is very difficult to observe experimentally. The reason is that it is very fragile with respect to the environment. For instance, if there is any disorder present, electrons prefer to sit at the local minima of the disorder

potential rather than at the lattice sites. It is difficult to create such low densities. Also, electrons are never isolated from the outside world — they were initially supplied by some atoms, and ions are left over. The potential created by these ions could also play a role and destroy the crystal. This is why three-dimensional Wigner crystal has never been observed.

Existing experimental observations of Wigner crystals come in two dimensions. One system is electrons above the surface of liquid Helium. There is no disorder, and this facilitates the observation of the Wigner crystallization. Also, Wigner crystal has been observed in semiconductor heterostructures, where one can create two-dimensional electron gas at the surface of the semiconductor. The concentration of two-dimensional electrons can be controlled by an external electrode — a gate. However, the Wigner crystal in these system have been observed only in strong magnetic fields. We will return to this below, when discussing the quantum Hall effect.





B. Quantum Hall effect

Another example of a strongly correlated system is two-dimensional electrons in a strong transverse magnetic field. Whether it is strongly correlated or not depends on the magnetic field. At some values of the field, associated with the *fractional quantum Hall effect*, indeed, new states of matter form. To understand this, we first need to go through the classical Hall effect in metals, and then discuss the integer quantum Hall effect, not associated with any strongly correlated systems.

Consider an electron in metal subject to an electric $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ fields. It experiences the force $e\boldsymbol{E}$ due to electric field and the Lorentz force $(e/c)\boldsymbol{v}\times\boldsymbol{B}$ due to magnetic field. We choose the z-axis parallel to the magnetic field and assume that there is voltage V applied in x-direction, and consequently there is current flowing in x-direction, but there is no current in y-direction. Since there is no current in the y-direction, there is no net force acting on electrons in this direction, and thus the y-component of the total force equals zero,

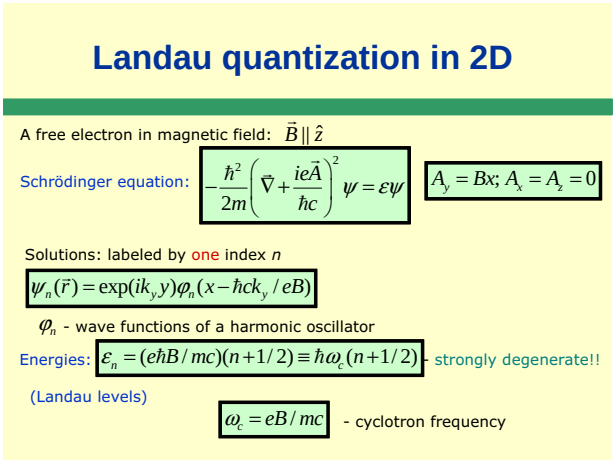
$$e\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)_y = 0\ .$$

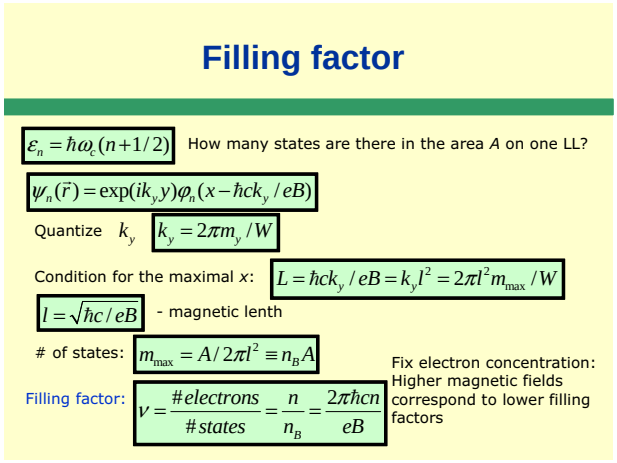
This means that electric field $E_y = Bv_x/c$ arises in y-direction. If the width of the system (dimension in y-direction) is W, the voltage $V_H = E_yW = BWv_x/c$ appears in the direction perpendicular to the current. This is known as *Hall effect*. This *Hall voltage* is proportional to the current. Indeed, the current density is $j = env_x$, where n is the concentration of electrons. In the case of interest, if the system is two-dimensional (electron can only move in x-y plane), the total current is $I = ev_xnW$, and thus $V_H = R_HI$, with

$$R_H = \frac{B}{enc} \tag{1}$$

known as *Hall resistance*. Note that the Hall resistance is proportional to the magnetic field. Usually it is much smaller than the ordinary longitudinal resistance. Another interesting feature is that the Hall effect, in contrast to the longitudinal resistance, is independent of the concentration of impurities. This independence persists up to rather strong fields. Indeed, an electron in magnetic field precesses around the direction of the field (in this case, in x–y plane) with the cyclotron frequency ω_c (see below). The above reasoning is correct as soon as $\omega_c \ll \tau^{-1}$, where τ is

electron scattering time associated with the impurities. In this regime, before an electron has a chance to complete the circle, it experiences many acts of impurity scattering. After each scattering event, the direction changes randomly, and thus it is not really important whether the electron trajectory between the impurities is a straight line of is slightly bend by the field. In contrast, in strong fields $\omega_c\tau \gg 1$ and electron completes many circles before being scattered. This considerably affects both longitudinal and Hall resistance.





Let us now turn to quantum mechanics and consider motion of a two-dimensional electron in x–y plane in perpendicular magnetic field B in z-direction. The electron wave function ψ is found from Schrödinger equation,

$$-\frac{\hbar^2}{2m}\left(\nabla+\frac{ie\boldsymbol{A}}{\hbar c}\right)^2\psi=\varepsilon\psi\ . \tag{2}$$

Here we have taken into account that in the presence of vector potential $\boldsymbol{A}$ the electron momentum $\boldsymbol{p}$ is replaced by the "long" momentum $\boldsymbol{p}-(e/c)\boldsymbol{A}$. The vector potential corresponds to the applied magnetic field, $\boldsymbol{B}=\nabla\times\boldsymbol{A}$. We choose it in the most convenient *Landau gauge*, $A_y = Bx$, $A_x = A_z = 0$. Note that the wave functions depend on the gauge, but the energy levels (which are observables) are gauge independent.

We have already encountered Eq. (2) in Lecture 5, when we discussed Landau diamagnetism (in three dimensions). Solutions in two dimensions are easily found. They are labelled by two indices, one continuous k_y and one discrete n,

$$\psi_n(k_y,\boldsymbol{r}) = e^{ik_y y}\varphi_n(x-\hbar c k_y/eB)\ , \tag{3}$$

where $\varphi_n(x)$ are the wave functions on one-dimensional Harmonic oscillator (decaying exponentials times the Hermite polynomials). In the y-direction we have a plane wave with the wave vector k_y, and in x-direction an oscillator wave function with the center shifted. The energy of this state only depends on n, not on k_y,

$$E=\hbar\omega_c(n+1/2),\quad \omega_c=\frac{eB}{mc}\ , \tag{4}$$

with the *cyclotron frequency* ω_c. Thus, the spectrum is a collection of discrete levels, corresponding to different values of n — *Landau levels*.

Landau levels are strongly degenerate — the same energy corresponds to a continuum of the values k_y. Let us look at this degeneracy more closely — calculate how many states are in a finite area A in one Landau level. Consider a system of a rectangular shape with the dimensions L in x-direction and W in y-direction. First, we quantize k_y by imposing periodic boundary conditions in the y-direction: $\psi(x,0)=\psi(x,W)$. This gives $k_y = 2\pi m_y/W$, m_y being positive integers. Furthermore, let us have a look at the wave functions. They only make sense as soon as the displacement of the oscillator center in the x-direction does not exceed L. But this displacement equals $\hbar c k_y/eB$, and thus the maximum value of m_y is achieved when this shift equals L,

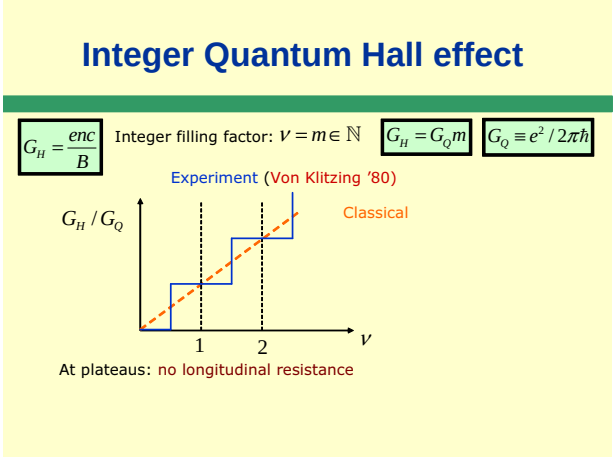
$$\hbar c k_y/eB < L \to m_{\max} = LW/2\pi l_B^2\ ,$$

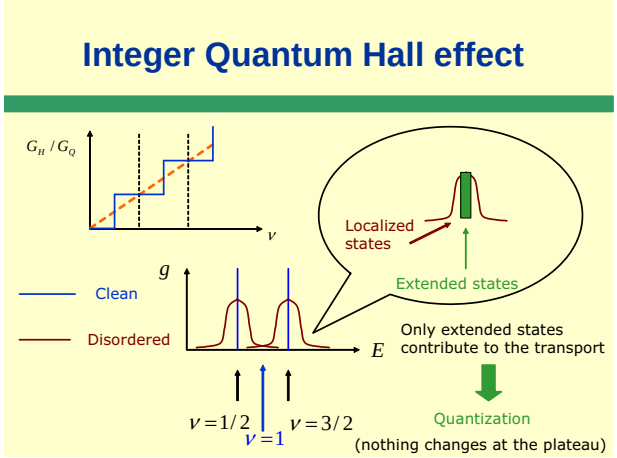
where we have introduced the *magnetic length* $l_B = (\hbar c/eB)^{1/2}$. Thus, the number of states at one Landau level in the system of the area A equals to $A/(\pi l_B^2)$, where we added a factor of 2 due to the spin degeneracy. This is the macroscopic degeneracy we expected to see: the number of states is proportional to the area. In other words, one electrons occupies an area of πl_B^2.

However, the number of electrons in the system can be controlled externally, for example, by the gate electrode. This means that not all available states are filled by the electrons. To express this notion, we introduce the *filling factor*,

$$\nu = \frac{\#\text{ of electrons}}{\#\text{ of states}} = \frac{n}{n_B} = \frac{\pi\hbar c n}{eB} ,$$

where n is the density of electrons, and $n_B = (\pi l_B^2)^{-1}$ is the number of states per unit area. For instance, if $\nu = 1$, we have all the states at the lowest Landau level ($n = 0$) occupied, and all states at higher Landau levels empty. For $\nu = 3$, all states in the three lowest Landau levels ($m = 0, 1, 2$) are occupied, and all states at higher Landau levels are empty. For $\nu = 1/2$, for instance, only one-half of the states at the lowest Landau level are occupied, and other states at the lowest Landau level and all states at higher Landau levels are empty. Note that the filling factor depends on the magnetic field. This means that we can keep the electron density constant, and different values of the field correspond to different filling factors. In high fields, the filling factor is low, and we are in the quantum regime. In very low fields, the filling factor is very high — many Landau levels are filled — and we are back to the classical regime, when the quantization is not important, and electrons can be treated as plane waves in both x and y directions.





In 1980, Klaus von Klitzing performed a pioneering experiment, for which he later received a Nobel Prize. He measured the conductance properties in a high magnetic field. The results are sketched at the Figure above. He measured both longitudinal and Hall resistance as the function of the magnetic field, and we present the results in form of the conductivity tensor σ_{ik}, defined as $j_i = \sigma_{ik} E_k$, i and k being the Cartesian indices x, y, z. The diagonal components $\sigma_{xx} = \sigma_{yy}$ describe the longitudinal resistance, whereas the off-diagonal components $\sigma_{xy} = -\sigma_{yx}$ describe Hall effect.

What von Klitzing discovered was something unexpected. Indeed, the off-diagonal (Hall) conductivity σ_{xy} as a function of the filling factor developed a number of plateaus around *integer* filling factor $\nu = n$. The transitions between the plateaus (around half-integer filling factors) were rather sharp. The value of σ_{xy} at the plateau with $\nu = n$ was $ne^2/(2\pi\hbar)$, *i.e.* determined by only a combination of fundamental constants. (The combination $e^2/(2\pi\hbar)$ occurs in many problems of quantum transport in nanostructures and is known as *conductance quantum*). The precision of the quantization was so good that the sensitivity to ten electrons in the whole system would already be observable. Measurements of quantized conductivity in magnetic field was later used as a metrological standard of conductance. The phenomenon of conductance quantization at the integer values of the filling factor is known as *Integer quantum Hall effect* (IQHE).

Another observation of the same experiment was that the longitudinal component of conductivity is absent at the plateaus. The longitudinal resistance only appears at the transitions between the plateaus, close to the half-integer filling factors.

It is important that this phenomenon can not be explained in terms of classical physics. Indeed, classically the Hall component of the conductivity is just the inverse Hall resistance provided the longitudinal resistance is zero, $\sigma_{xy} = R_H^{-1} = enc/B = e^2\nu/\hbar$ — proportional to the filling factor. Indeed, precisely in the middle of the plateaus the Hall conductivity measured in the experiment corresponds to the classical expressions. However, classical Hall effect does not exhibit any plateaus as the function of the filling factor.

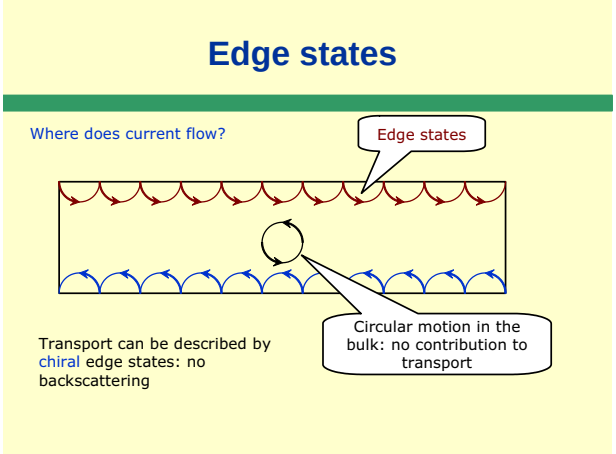
To explain the appearance of the plateaus, one needs to involve impurities. In the presence of disorder, the energy of electrons is not precisely quantized any more. The wave function (3) acquire a finite lifetime τ due to the scattering.

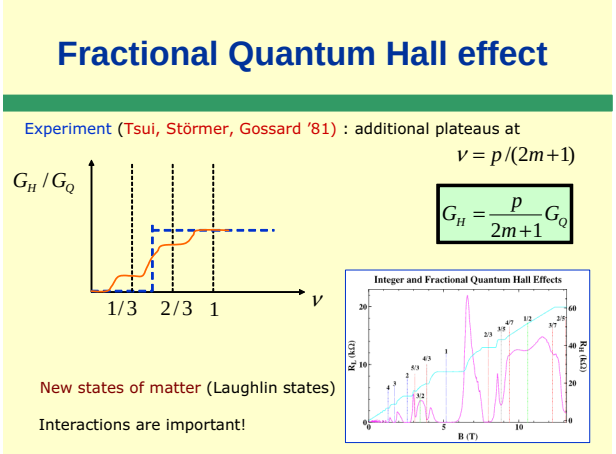
The Heisenberg uncertainty relation implies that the Landau levels acquire a finite width $\Gamma \sim \hbar/\tau$ — the energy of electrons are not precisely $\hbar\omega_c(n+1/2)$ but lie in the vicinity of these values. If we plot the density of states vs energy, in the absence of impurities it represents a set of delta-functions located at the Landau level positions. Impurities smear these delta-functions into (Lorentzian) peaks of the finite width of the order Γ. Let us look at a single peak in details. The electron states do not resemble plane waves any more. Still, some of them are extended through the whole sample. It turns out that these states have energies in the middle of a Landau level. Other states, which are at the tails of the peak, are localized — the wave function is confined to a small region of space. If on increases the filling factor, new states get filled. For instance, if the states are filled at the first Landau level, for $\nu < 1/2$ mainly localized states are filled, and other states are empty. In the vicinity of $\nu = 1/2$ extended states are being filled, and for $1/2 < \nu < 1$ localized states with energies above $\hbar\omega_c/2$ get filled. At $\nu = 1$ the first Landau level is full, and the states of the second Landau level are all empty.

However, the transport (both longitudinal and Hall resistance) is only determined by extended states — localized states can not transport charge through the whole system. This is why the Hall resistance only varies in the vicinity of half-integer filling factors, where the extended states are being filled, and the longitudinal resistance vanishes between the center of Landau levels.

We explained the existence of the plateaus of the hall conductivity, but these arguments are not sufficient to explain the value of the Hall conductivity at the plateaus. For this one would need advanced arguments which go beyond the scope of the course.

Note that even though there is no longitudinal conductivity in the bulk of two-dimensional electron gas at integer filling factors, one can still transport. Indeed, every sample has finite dimensions. For $\nu = 1$ (at the plateau) all filled states are localized - the electrons perform finite motion. For instance, if impurities are sparse, electrons classically move counterclockwise in circles — cyclotron motion in the plane perpendicular to the magnetic field. If the circle — the cyclotron orbit — lies inside the sample, such an electron does not participate in transport. However, if the center of the orbit is close to the edge of a rectangular sample, the orbit can not close. In this case, and electrons move in so called *skipping orbits* — pieces of the circles that are partially closed and partially reflected from the boundary. It is important that the net direction of electron motion is fixed by the direction of magnetic field: The electrons can only move right along the upper edge and left along the lower edge. In quantum mechanics, one can not talk about classical trajectories and skipping orbits — instead, one talks of edge channels. Edge channels are chiral: the electron can only move in a fixed direction, scattering back in impossible.





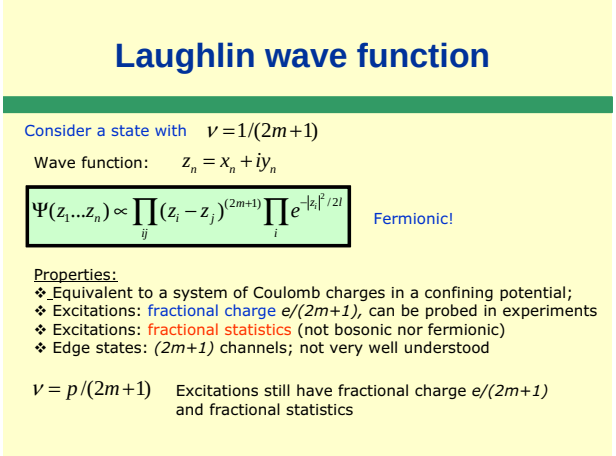
In 1981, Daniel Tsui and Horst Störmer performed another experiment. They were hoping to refine the findings of von Klitzing, and indeed they discovered something unexpected — besides the plateaus at integer filling factors, also the plateaus at the rational value of the filling factor with an odd denominator appeared. The bigger was the denominator, the better were the plateaus pronounced. The best were the plateaus at $\nu = 1/3$ and $\nu = 2/3$, then $\nu = 2/5$ and $\nu = 3/5$, and, for instance, the plateau at $\nu = 4/7$ is hardly visible. The Hall conductivity at the center of the plateau corresponding to the filling factor ν is $\nu e^2/(2\pi\hbar)$. This phenomenon is known as *fractional quantum Hall effect* (FQHE). For this discovery, Tsui and Störmer eventually also received a Nobel Prize, which they shared with Laughlin.

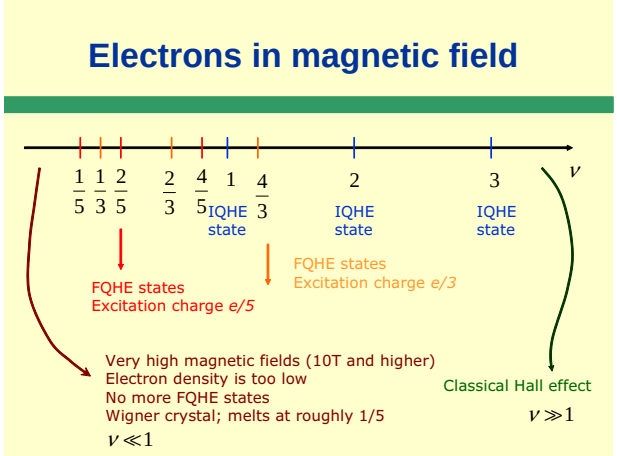
Obviously, plateaus at rational values of ν can not be explained in the model of free electrons, even if they scatter at impurities. Thus, FQHE is due to electron-electron interactions. New states of matter (Laughlin states) form at the values of the filling factor $\nu = p/(2m+1)$, m and p being integers. To describe the ground state with $\nu = 1/(2m+1)$, Robert Laughlin suggested the *multi-particle* wave functions, which depends on the coordinates of

all electrons present in the system, $\boldsymbol{r}_1 \dots \boldsymbol{r}_N$. This Laughlin function is best written in terms of complex variables $z_n = x_n + iy_n$, $n = 1 \dots N$,

$$\Psi(z_1 \dots z_N) \propto \prod_{ij} |z_i - z_j|^{(} 2m+1) \prod_i e^{-|z_i|^2/2l_B} \ . \tag{5}$$

Note that this function is fermionic — it changes sign if any two particles are interchanged. This properties is ensured by the fact that the denominator $2m+1$ is odd. For an even denominator, we would not be able to write an analog of the Laughlin function.





There are no rigorous arguments which could help us to *derive* the Laughlin function. The initial motivation was that similar functions well describe states in liquid helium, which may sound somewhat weak. However, the results of numerical diagonalization of a system of interacting electrons in strong magnetic field showed that, even though Eq. (5) apparently does not describe the ground state of the electron system, the ground state energy it provides is within the 4 percent agreement with the exact result. It means that the Laughlin function describes the ground state pretty much accurately.

Note that Eq. (5) has a physical interpretation. Indeed, imagine that Ψ is the partition function of classical particles. The corresponding free energy $F = -k_B T \ln \Psi$ reads

$$F - F_0 = k_B T \sum_i \frac{|z_i|^2}{2l_B} - k_B T (2m+1) \sum_{ij} \ln |z_i - z_j| \ .$$

If we interpret this as free energy of classical particles, the first term in the rhs describes the motion of the particles in parabolic confining potential, and the second one represents logarithmic interactions between the particles (the strength of the interaction is proportional to $2m+1$). Note that in two dimensions, Coulomb interactions are logarithmic. Thus, the ground state of FQHE phases is equivalent to the ground state of a system of two-dimensional Coulomb particles in a parabolic potential.

The system with the ground state (5) has a number of unusual properties which we list here. The *excited states* on top of the Laughlin ground state have fractional charge $e/(2m+1)$. This fractional charge has been measured in the shot noise experiments. These excitations also obey fractional statistics — they are neither bosons nor fermions. It is very difficult to design an experiment sensitive to the statistics of quasiparticles, and this statistics has not yet been measured. The edge states have $2m+1$ transport channels.

For more complicates states, $\nu = p/(2m+1)$, the excitations have the charge $e/(2m+1)$ (not $ep/(2m+1)$) and also obey the fractional statistics.

Let us now summarize the behavior of two-dimensional electrons in perpendicular magnetic field. Let us start with low fields and then increase the field; this corresponds to the gradual decrease in the filling factor. High values of ν correspond to the classical system — for $\nu \gg 1$ the distance between Landau levels is small, and quantum Hall effect is not observed. For lower values of the filling factor, IQHE states (integer ν) and FQHE states (rational ν with an odd denominator) start to proliferate. At very low filling factors (corresponding to the field 10 T and higher), $nu < 1/5$, no FQHE states are observed. At these filling factors, the electron density is so low that Wigner crystal is the preferred state of the system. Wigner crystallization in high magnetic fields has been observed experimentally.

C. Kondo effect

(Will be added later; in 2007 is not a part of the course).