D & S
AE3-914

March 28, 2011

Calculus of variations

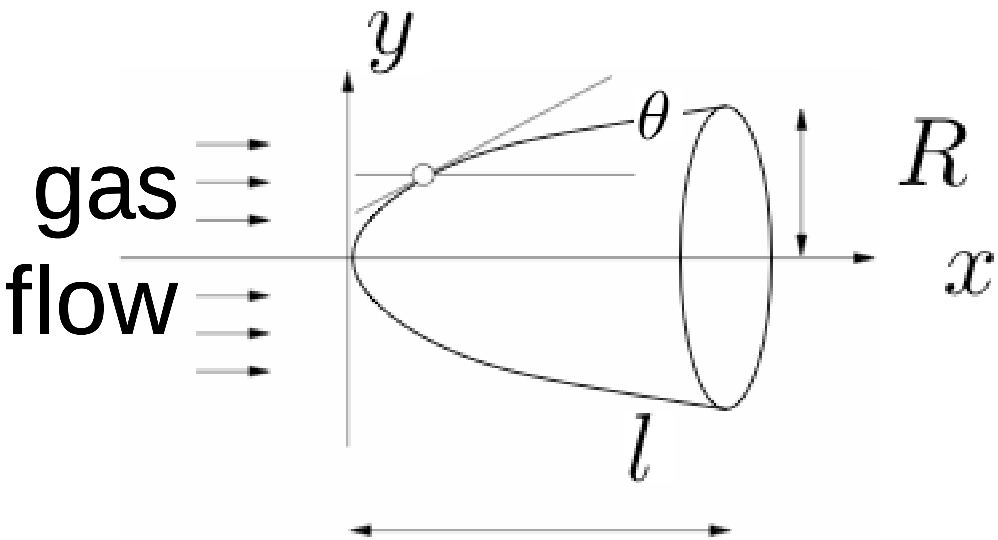
Find the function *y(x)* that minimizes a functional

$$I(y) = \int_{x_a}^{x_b} F(x, y, y')\, dx$$

Euler-Lagrange equation

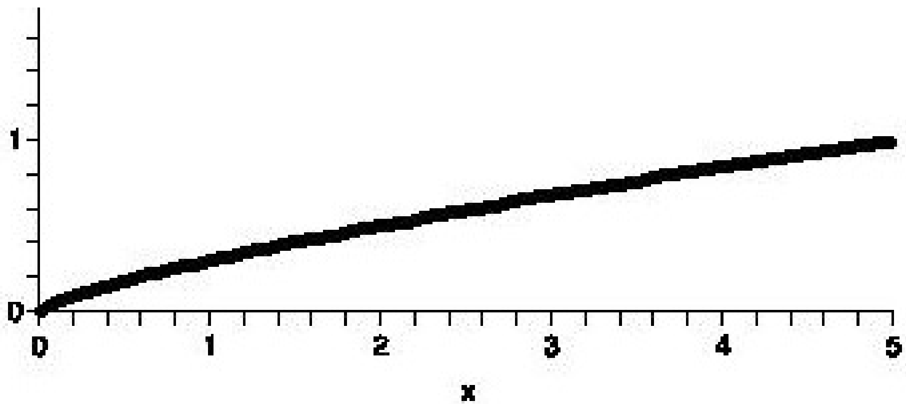
$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Example:



Shape *y(x)* for minimal resistance (i.e., minimal dynamic pressure)?

(assume axial symmetry about *x)*



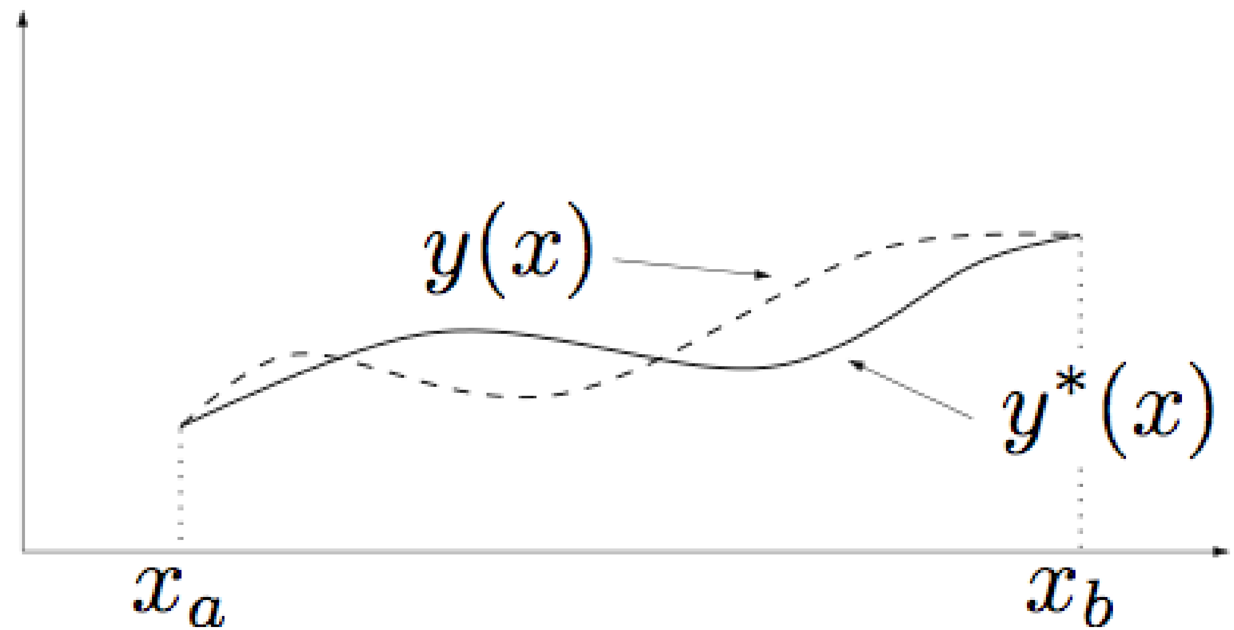
$$l = 5\,\mathrm{m}$$

$$R = 1\,\mathrm{m}$$

$$y = R\left(\frac{x}{l}\right)^{3/4}$$

(R= the radius at x=l)

For functionals one needs to define what is understood as a variation



Variational operator

$$
\begin{aligned}
y(x) &= y^*(x) + \varepsilon\eta(x) \\
&= y^*(x) + \delta y^*(x)
\end{aligned}
$$

$$I(y) = I(y^* + \delta y^*) = I(y^*) + \delta I$$

Function *y* provides an extremal of *I* if

$$\delta I = 0$$

"δ" is an infinitesimal change (or variation)

$$\begin{aligned}\delta I &= \int_{x_a}^{x_b} \delta F(x, y, y')\, dx \\ &= \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right)\, dx\end{aligned}$$

$$= \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} \delta y - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \delta y \right) dx + \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_a}^{x_b}$$

Since δy represents any variation of y

and $\delta y(x_a) = \delta y(x_b) = 0$ we end again with

Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Natural boundary conditions

$$\delta I = \int_{x_a}^{x_b} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right] \delta y \, dx + \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_a}^{x_b}$$

where the *essential boundary condition* is

$y(x_a) = y_a$, but nothing is said about $y(x_b)$

Since $\delta y(x_b) \neq 0$ we must have

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_b} = 0$$

which is a *natural boundary condition*

Examples of boundary conditions

Continuum:

Prescribing traction (*natural* B.C.)

Prescribing displacement (*essential* B.C.)

Beam:

Prescribing bending moment/shear force (*natural* B.C.)

Prescribing displacement/rotation (*essential* B.C.)

Generalisations of the Euler-Lagrange equations

$$I(\mathbf{y}) = \int_{x_a}^{x_b} F(x, \mathbf{y}, \mathbf{y}')\, dx \qquad \mathbf{y} \in \mathbb{R}^n$$

with the vector of independent functions: $\mathrm{y} = (y_1 , y_2 , \ldots , y_n)$

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) = 0 \qquad i = 1, \ldots, n$$

$$I(y) = \int_{x_a}^{x_b} F(x, y, y', y'', \dots, y^{(n)})\, dx$$

$$\frac{\partial F}{\partial y} + \sum_{i=1}^{n} (-1)^i \frac{d^i}{dx^i} \left(\frac{\partial F}{\partial y^{(i)}} \right) = 0$$

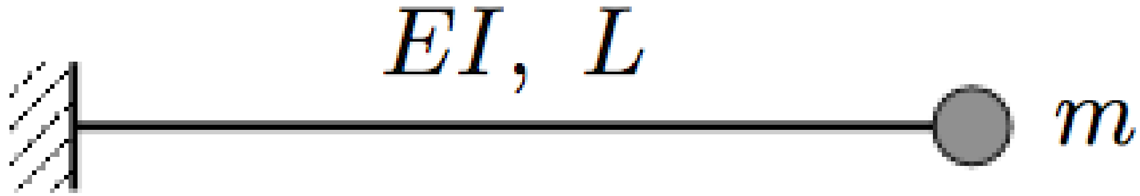
$$I(y) = \int_{t_a}^{t_b} \int_{x_a}^{x_b} F(x, t, y, y_x, y_t)\, dx dt$$

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) = 0$$

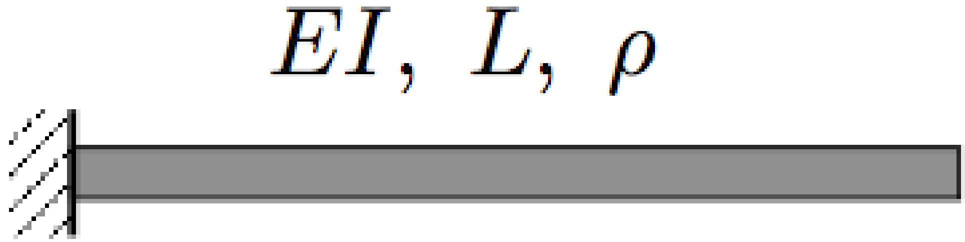
$$I(u) = \int_{t_a}^{t_b} \int_{x_a}^{x_b} F(x, t, u, u_t, u_{xx}) \, dx dt$$

where $\quad u = u(x, t), \; u_t = \dfrac{\partial u}{\partial t}, \; u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$

Euler-Lagrange equation?



$$m\ddot{u} + \frac{3EI}{L^3}u = 0$$



Equations of motion?

Hamilton's principle

The motion of a Lagrangian system is such that

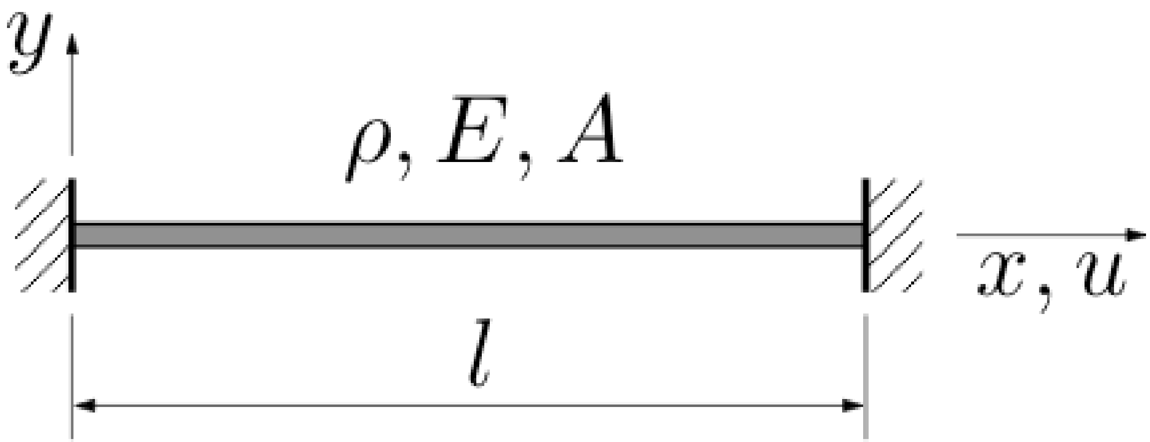
$$\int_{t_a}^{t_b} (T - V)\, dt \quad \text{is a minimum}$$

Hamilton’s principle

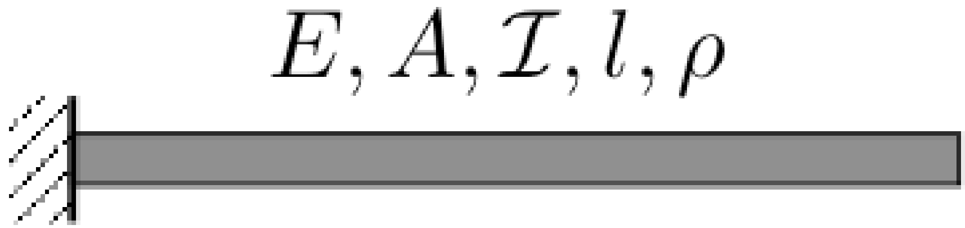
$$\delta \int_{t_a}^{t_b} (T - V)\, dt = \int_{t_a}^{t_b} \delta L\, dt = 0$$

If the system is described by a set of *n* generalised coordinates **q**

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \qquad k = 1, \ldots, n$$



Equations of motion for $u(x,t)$?



Equations of motion?

Boundary conditions for $x = 0$ and $x = l$?

Non-conservative forces

$$\delta W = \delta W^{c} + \delta W^{nc}$$

$$= -\frac{\partial V}{\partial q}\delta q + Q^{nc}\delta q$$

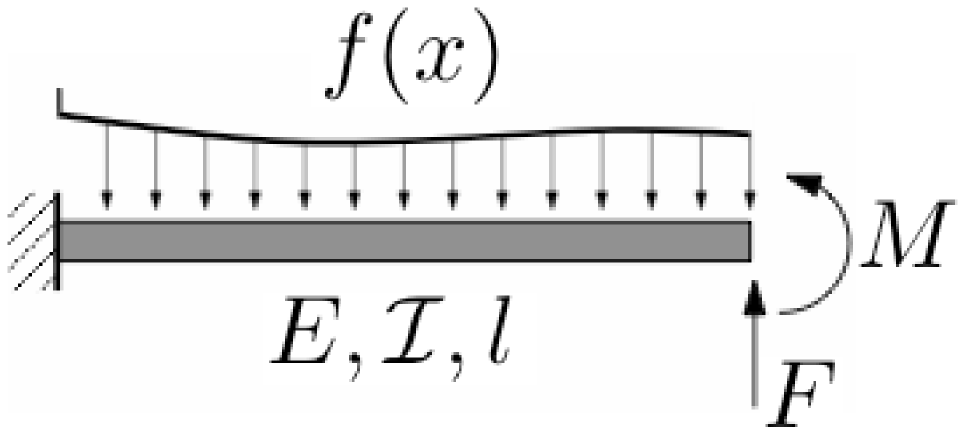
$$= -\delta V + Q^{nc}\delta q$$

$$\delta I(\mathbf{q}) = \int_{t_a}^{t_b} \left(\delta L + \delta W^{nc} \right) dt = 0$$

with $$\delta W^{nc} = \sum Q_i^{nc} \delta q_i$$

Statics: Hamilton's Principle reduces to
Principle of Stationary Potential Energy

$$\delta V = 0$$



Equilibrium equation and boundary conditions?