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5. MODELING OF NON-STRATIFIED MIXTURE FLOWS (Pseudo-Homogeneous Flows)

NEWTONIAN SLURRIES

NON-NEWTONIAN SLURRIES
Repetition: Newtonian vs non-Newtonian

Newton’s law of liquid viscosity (valid for laminar flow)

\[ \tau = \mu_f \left( -\frac{dv_x}{dr} \right) \]
Repetition: Newtonian vs non-Newtonian

Newton’s law of liquid viscosity
(valid for laminar flow):

\[ \tau = \frac{F}{A} = \mu_f \left( - \frac{dV_x}{dy} \right) \]
MODELING OF NEWTONIAN SLURRY FLOW

EQUIVALENT-LIQUID MODEL
Equivalent-Liquid Model

A. Physical background

- Slurry flow behaves as a flow of a single-phase liquid having the density of the slurry.

- The "equivalent liquid" has the density of the mixture but other properties (as viscosity) remain the same as in the liquid (water) alone.
Equivalent-Liquid Model

B. Construction of the model for $I_m$:

- The model suggests that all particles contribute to the increase of the suspension density.
- The increase of the mixture density is responsible for the increase of the liquid-like shear stress resisting the flow at the pipe wall.
- The model equation is obtained in the same way as the Darcy-Weisbach equation with one exception: the density of mixture is considered instead of the liquid density.
Repetition: Conservation of Momentum in 1D-flow

For additional conditions:
- incompressible liquid,
- steady and uniform flow in a horizontal straight pipe

\[- \frac{dP}{dx} A = \tau_o O, \quad \text{i.e.} \quad - \frac{dP}{dx} = \frac{4\tau_o}{D}\]

for a pipe of a circular cross section and internal diameter D.
Repetition: Water Friction in 1D Flow in Pipe

A comparison of the Darcy-Weisbach friction coefficient equation

\[ \lambda_f = \frac{8\tau_o}{\rho_f V_f^2} \]

with the linear momentum eq. (driving force = resistance force) for pipe flow

\[ \frac{dP}{dx} = \frac{4\tau_o}{D} \]

= gives the general pressure-drop equation for the pipe flow (Darcy-Weisbach equation, 1850)

\[ \frac{dP}{dx} = \frac{\lambda_f \rho_f V_f^2}{D^2} \]
Equivalent-Liquid Model

B. Construction of the model for $I_m$ (cont’):

- The shear stress at the pipe wall:

  \[
  \tau_{o,f} = \frac{\lambda_f}{8} \rho_f V^2
  \]
  
  \[
  \tau_{o,m} = \frac{\lambda_f}{8} \rho_m V^2
  \]

- The hydraulic gradient for slurry (equiv.-liquid) flow, $I_m$, is

  \[
  I_m = -\frac{dP}{dx \rho_f g} = \frac{\lambda_f}{D} \frac{V_m^2}{2g \rho_f} \rho_m = I_f \frac{\rho_m}{\rho_f}
  \]
Very fine sand (MTI) in 158-mm horiz pipe

Im - Vm curves for viscous ELM model

(x): water
(*): $C_{vd} = 0.18$
(): $C_{vd} = 0.24$
(+): $C_{vd} = 0.30$
(*): $C_{vd} = 0.36$
(o): $C_{vd} = 0.42$

Hydraulic gradient Im [-]
Mean mixture velocity Vm [m/s]
Fine sand (DUT) – vertical flow

(+): water
(*): $C_{vd} = 0.10$
(Δ): $C_{vd} = 0.14$
(◊): $C_{vd} = 0.21$
(∇): $C_{vd} = 0.27$
(): $C_{vd} = 0.34$

Hydraulic gradient [-]

Mean mixture velocity [m/s]
Fine sand (DUT) – vertical flow

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(*): $C_{vd} = 0.10$

(Δ): $C_{vd} = 0.14$

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(): $C_{vd} = 0.34$

Mean mixture velocity [m/s] vs. Hydraulic gradient [-]
Generalized Equivalent-Liquid Model

Model for both fine and coarse pseudo-homogeneous flows (e.g. vertical flows):
The hydraulic gradient for slurry flow, $I_m$, is

$$\frac{I_m - I_f}{C_{vd} (S_s - 1)} = A'I_f$$

If $A'=1$, then $I_m = S_m I_f$, i.e. the equivalent-liquid model.
If $A'=0$, then $I_m = I_f$, i.e. the liquid (water) model.
If $0 < A' < 1$, then $I_m$ according to the generalized model with calibrated $A'$ value.
MODELING OF NON-NEWTONIAN SLURRY FLOW

MANY DIFFERENT MODELS
Non-Newtonian Slurry Model

A. Physical background

- Very fine particles increase the viscosity of the pseudo-homogeneous mixture.

- The suspension does not obey Newton’s law of viscosity and its constitutive rheological equation has to be determined experimentally. Each particular mixture obeys its own law of viscosity.

- Modeling of non-Newtonian mixtures is even more complex than the modeling of Newtonian mixtures in pipes.
Non-Newtonian Slurry Model

A. Physical background

- Internal structure of flow compared with Newtonian:
  - different velocity distrib.
  - different viscosity distrib.
  - identical shear-stress distribution
Non-Newtonian Slurry Model

B. Construction of the model for $I_m$
A general procedure for a determination of the frictional head loss, $I_m$:

1. the **rheological parameters** of a slurry

2. the slurry **flow regime** (laminar or turbulent)

3. the **pressure drop** using a scale-up method or an appropriate friction model.
Repetition: Newtonian vs non-Newtonian

Newton’s law of liquid viscosity (valid for laminar flow)

\[ \tau = \mu f \left( -\frac{dv_x}{dr} \right) \]
## Step 1. Rheological Parameters

### Table. Typical values of dynamic viscosity $\mu$ at room temperature

<table>
<thead>
<tr>
<th>Substance</th>
<th>Dyn. Viscosity $\mu$ [mPa.s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>Water</td>
<td>1</td>
</tr>
<tr>
<td>Olive oil</td>
<td>$10^2$</td>
</tr>
<tr>
<td>Honey</td>
<td>$10^4$</td>
</tr>
<tr>
<td>Molten glass</td>
<td>$10^{15}$</td>
</tr>
</tbody>
</table>

*(source: Chhabra & Richardson, 1999)*
Step 1. Rheological Parameters

Rotational viscometer

- Pointer
- Retaining spring
- Rotating outer cylinder
- Stationary inner cylinder
Step 1. Rheological Parameters

Viscometers for determination of rheological parameters:

Rotational Capillary

Dredge Pumps and Slurry Transport
Step 1. Rheological Parameters

Rheograms

from rotational viscometers:

\[ \tau = fn \left( -\frac{dv_x}{dr} \right) \]

from tube (capillary) viscometers:

\[ \tau = fn \left( -\frac{dv_x}{dy} \right) \]
**Step 1. Rheological Parameters: Tube Viscometer**

*Conditions*:
- incompressible and homogeneous slurry,
- steady and uniform flow in horizontal straight tube,
- laminar flow (will be required in next slide)

\[
- \frac{dP}{dx} = \frac{4\tau_0}{D} \implies \tau_0 = \frac{\Delta P}{L} \frac{D}{4}
\]

The wall shear stress \(\tau_0\) obtained from \(-dP/dx = \Delta P/L\) measured in a tube of the internal diameter \(D\) over the length \(L\).
Step 1. Rheological Parameters: Tube Viscometer

*Conditions*:
- incompressible and homogeneous slurry,
- steady and uniform flow in horizontal straight tube,
- laminar flow $\lambda_m = \frac{64}{Re}$ !!!

\[
\tau_o = \frac{\lambda_m}{8} \rho_m V_m^2 \quad \text{and} \quad \lambda_m = \frac{64}{Re} = \frac{64 \mu_m}{V_m D \rho_m}
\]

Combination of these two equations gives the relationship representing the **pseudo-rheogram** $\tau_0 = \text{fn}(V_m/D)$ obtained from the tube viscometer.
Step 1. Rheological Parameters: Tube Viscometer

**Conditions**:
- incompressible and homogeneous slurry,
- steady and uniform flow in horizontal straight tube,
- laminar flow

\[
\tau_o = \mu_m \frac{8V_m}{D} \quad \text{(pseudo)equivalent to } \tau = \mu_m \left( -\frac{dv_x}{dy} \right)
\]

The method must be found that transforms a pseudo-rheogram to a real rheogram.
The Rabinowitsch-Mooney transformation transforms:
- the pseudo-rheogram \([T_0, 8V_m/D]\) to
- the normal rheogram \([T, dv_x/dy]\) and vice versa

for a laminar flow of non-Newtonians.
Intermezzo. Rabinowitsch-Mooney Transformation

Procedure:

1. The pseudo-rheogram: determined experimentally in a tube viscometer: points \([T_o, 8V_m/D]\) plotted in the ln-ln co-ordinates give

\[\tau_o = K \left( \frac{8V_m}{D} \right)^{n'}, \quad \text{where} \quad n' = K \frac{d \left( \ln \tau_o \right)}{d \left( \ln \frac{8V_m}{D} \right)}\]

2. Transformation rules: \(T = T_o\) and \(\frac{dv_x}{dy} = \left( \frac{3n' + 1}{4n'} \right) \frac{8V_m}{D}\)
   (for Newtonians \(n' = 1\))

3. The normal rheogram: the \([T, dv_x/dy]\) points are fitted by a suitable rheological model to determine the rheol. parameters
Step 1. Rheological Parameters

**Pseudo-rheograms** by capillary or tube viscometers:
- wall shear stress, $T_0$
- mean velocity, $V_m$

\[
T_0 = 35.5 \left( \frac{8V_m}{D} \right)^{0.11}
\]
Step 1. Rheological Parameters

Transformed to the normal rheograms using the Rabinowitsch-Mooney transformation:

- shear stress, $T$
- shear rate, $\frac{dv}{dy}$

Result: the slurry behaves as a Bingham plastic liquid, its internal friction is described by two rheological parameters, $T_y, \eta_B$. 
Step 1. Rheological Parameters

Rheograms of various non-Newtonian slurries:

- Dilatant (2 parameters)
- Pseudo-plastic (2 parameters)
- Bingham (2 parameters)
- Yield dilatant (3 parameters)
- Yield pseudo-plastic (3 parameters)
Step 1. Rheological Parameters

Example of rheogram obtained from viscometer test

Fig. 7.1 Rheogram for clay slurry.
Step 1. Rheological Parameters

**Rheograms** of various non-Newtonian slurries:

- Dilatant (2 parameters)
- Pseudo-plastic (2 par.)
- Bingham (2 par.)
- Yield dilatant (3 parameters)
- Yield pseudo-plastic (3 par.)
Step 1. Rheological Parameters

Rheological model with two-parameters

The Bingham-plastic model contains rheological parameters $\tau_y$, $n_B$. The rheogram is a straight line with the slope $n_B$ (called plastic viscosity or tangent viscosity). $\tau_y$ is called the yield stress.

$$\tau = \tau_y + \eta_B \frac{dv_x}{dy}$$

This model satisfies the flow behaviour of the majority of fine homogeneous high-concentrated slurries that are dredged (aqueous mixtures of non-cohesive clay etc.).
Step 1. Rheological Parameters

Rheological model with two-parameters

The Power-law model contains rheological parameters \(K, n\). The rheogram is a curve. \(n\) is called the flow index (it is not the viscosity). \(K\) is called the flow coefficient.

\[
\tau = K \left( \frac{dv_x}{dy} \right)^n
\]

Different types of fluid behavior occur:
- \(0 < n < 1\): pseudo plastic slurry (becomes thinner under the increasing shear rate)
- \(n > 1\): dilatant slurry (becomes thicker under the increasing \(dv/\text{dy}\)).
Rheological model with three-parameters

The Yield Power-law model (Buckley-Herschel model) contains rheological parameters $T_\gamma$, $K$, $n$. The rheogram is a curve which does not pass through origin.

$$\tau = \tau_y + K \left( \frac{dv_x}{dy} \right)^n$$

Different types of fluid behavior occur:

* $0 < n < 1$: pseudo plastic slurry (becomes thinner under the increasing shear rate)
* $n > 1$: dilatant slurry (becomes thicker under the increasing $dv/dy$).
Threshold for Bingham plastic slurries is given by the value of the modified Bingham Reynolds number

\[ \text{Re}_B = \frac{\rho_m V_m D}{\eta_B \left( 1 + \frac{\tau_y D}{6 \eta_B V_m} \right)} = 2100 \]

in which \( \tau_y \) and \( \eta_B \) are the rheological parameters of the Bingham plastic slurry.
Step 3. Pressure drop due to friction

Two Approaches:

Scale Up
or
Friction model.

There are different scale-up methods for laminar and turbulent flows. There are different friction models for laminar and turbulent flows.
Step 3. Pressure drop: Scale Up

**Principle**

The $I_m - V_m$ measurement results from tube viscometers can be scaled up to prototype pipes without an intermediary of a rheological model (for the flow of identical slurry).

The principle of the scaling-up technique is that in non-Newtonian flows the wall shear stress is unchanged in pipes of different pipe sizes $D$. 
Step 3. Pressure drop: Scale Up

Laminar flow
using the Rabinowitsch-Mooney transformation:

\[ \tau_o = \frac{D \cdot \Delta P}{4L} = \text{idem} \]
\[ \frac{8V_m}{D} = \text{idem} \]

This says that if a \( \tau_o \) versus \( 8V_m/D \) relationship is determined experimentally for a laminar flow of certain slurry in one pipe it is valid also for pipes of all different sizes.

Scaling up between two different pipeline sizes [e.g. between a tube viscometer (index 1) and a prototype pipeline (index 2)]:

\[ I_{m2} = I_{m1} \frac{D_1}{D_2} \]
\[ V_{m2} = V_{m1} \frac{D_2}{D_1} \]
Step 3. Pressure drop: Scale Up

Turbulent flow
The Rabinowitsch-Mooney transformation not applicable:

\[ \tau_o = \frac{D_1 \Delta P}{4L} = \text{idem} \]

\[ \frac{8V_m}{D_1} \neq \text{idem} \]

because the near-wall velocity gradient is not described by \(8V_m/D_1\), instead the wall-friction law uses an empirical friction coefficient to relate the wall shear stress with the mean velocity.

Scaling up between two different pipeline sizes [e.g. between a tube viscometer (index 1) and a prototype pipeline (index 2)]:

\[ I_{m2} = I_{m1} \frac{D_1}{D_2} \]

\[ V_{m2} = V_{m1} \left[ 1 + 2.5 \sqrt{\frac{\lambda_f}{8}} \ln \left( \frac{D_2}{D_1} \right) \right] \]
Step 3. Pressure drop: Scale Up

Turbulent flow
The example of the scale up of the turbulent flow of slurry:

- data *measured* in the 77-mm pipe
- and *predicted* for the 154-mm pipe.

![Graph showing shear stress at boundary vs. 8V_m/D (s⁻¹)]
Step 3. Pressure drop: Friction Model

Laminar flow

Integrating of a rheological model over a pipe cross section gives the mean velocity and the relation between $V_m$ and the pressure drop (the same procedure as for the water flow).

The obtained equation can be rearranged to the standard eq.

$$I_m = \frac{\Delta P}{\rho_f g L} = \frac{\lambda_{nN}}{D} \frac{V_m^2}{2g}$$

in which the wall-friction law for the laminar flow: and thus all effect on the frictional pressure drop due to slurry rheology are expressed in the modified Reynolds number.

$$\lambda_{nN} = \frac{64}{\text{Re}_{nN}}$$
Step 3. Pressure drop: Friction Model

Laminar flow – EXAMPLE: Bingham plastic slurry flow

The rheological model

\[
\tau = \tau_y + \eta_B \frac{dv_x}{dy}
\]

after integrating

\[
\frac{8V_m}{D} = \frac{\tau_o}{\eta_B} \left(1 - \frac{4\tau_y}{3\tau_o} + \frac{\tau_y^4}{3\tau_o^4}\right)
\]

compared with the basic force-balance and Darcy-W. equations

\[
\frac{\Delta P}{L} = \frac{4\tau_o}{D}
\]

and

\[
I_m = \frac{\Delta P}{\rho_f g L} = \frac{\lambda_{nN} V_m^2}{D 2g}
\]

leads to the wall-friction law for the Bingham-slurry flow =>
Step 3. Pressure drop: Friction Model

Laminar flow – EXAMPLE: Bingham plastic slurry flow

leads to the wall-friction law for the Bingham-slurry flow

\[
\lambda_{nN} = \frac{64\eta_B}{DV_m \rho_m \left(1 - \frac{4\tau_y}{3\tau_o} + \frac{\tau_y^4}{3\tau_o^4}\right)} = \frac{64}{Re_B}
\]

The modified Bingham Reynolds number is

\[
Re_B = \frac{\rho_m V_m D}{\eta_B \left(1 + \frac{\tau_y D}{6\eta_B V_m}\right)}
\]

(the fourth-power term in the above eq. is neglected)
Step 3. Pressure drop: Friction Model

Turbulent flow
In turbulent flow, a velocity profile differs from that determined by a rheological model (analogy with a Newtonian flow).

The standard Darcy-Weisbach equation is valid

\[ I_m = \frac{\Delta P}{\rho_f g L} = \frac{\lambda_{nN} V_m^2}{D \ 2g} \]

but the wall-friction law is an empirical function

\[ \lambda_{nN} = f_n(Re_{nN}, k, \ldots) \]
Step 3. Pressure drop: Friction Model

Turbulent flow - EXAMPLE
The most successful models for turbulent flows of non-Newtonian slurries are:

Wilson – Thomas model

Slatter model.