Ritz method for approximate solution of a variational problem

Miguel A. Gutiérrez
Engineering Mechanics Group, Faculty of Aerospace Engineering, Delft University of Technology

Ritz method is an approximative technique to find the solution of a variational problem. Consider, for example, a variational problem in the form

$$I[y(x)] = \int_{x_a}^{x_b} F(x, y(x), y(x)')\, dx; \qquad y(x_a) = y_a; \quad y(x_b) = y_b. \tag{1}$$

An extremal would be found as the solution of Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{2}$$

with the essential boundary conditions given in (1).

Ritz method is searching a solution to the variational problem on the functional (1) directly, rather than on the differential equation (2). The basic idea is to approximate the solution function $y(x)$ as a linear combination of n *known* functions $h_i(x)$

$$\bar{y}(x) = \sum_{i=1}^{n} a_i h_i(x), \tag{3}$$

where the notation $\bar{y}$ indicates that (3) is not the exact solution but a mere approximation and the coefficients a_i are *unknown* scalars. These scalars are determined by substituting the linear combination (3) into the functional (1). Notice that since the functions $h_i(x)$ are known so are their derivatives. The derivative of $\bar{y}(x)$ can therefore be written as

$$\bar{y}'(x) = \sum_{i=1}^{n} a_i h_i'(x). \tag{4}$$

Substituting (3) and (4) into (1), the functional depending on $y(x)$ is converted into a function Φ of the n variables a_i,

$$\Phi(a_1, \dots a_n) = \int_{x_a}^{x_b} F\left(x, \sum_{i=1}^{n} a_i h_i(x), \sum_{i=1}^{n} a_i h_i'(x)\right) dx. \tag{5}$$

Indeed, the functions $h_i(x)$ are known and the integration can be carried out with respect to the variable x. Similarly, the boundary conditions are expressed as

$$\sum_{i=1}^{n} a_i h_i(x_a) = y_a; \quad \sum_{i=1}^{n} a_i h_i(x_b) = y_b, \tag{6}$$

which are linear expressions in the variables a_i. The variational problem has been reduced to finding the coefficients a_i for which the function (5) attains an extreme value, subject to the restrictions (6). This can, e.g., be done by substituting the restrictions (6) into (5), differentiating (5) with respect to the remaining variables a_i and imposing that these derivatives are equal to zero.

The functions $h_i(x)$ and the coefficients a_i are often referred to as *shape functions* and *degrees-of-freedom* respectively. The shape functions must be linearly independent, i.e.,

$$\sum_{i=1}^{n} a_i h_i(x) \equiv 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0. \tag{7}$$

Examples of sets of shape functions fulfilling this property would be, for $n = 3$,

$$h_1(x) = 1; \quad h_2(x) = x; \quad h_3(x) = x^2, \tag{8}$$

that is, polynomials of zero, first and second degree, but also a set of second-degree polynomials only such as

$$h_1(x) = (x - x_a)^2; \quad h_2(x) = (x - x_a)(x - x_b); \quad h_3(x) = (x - x_b)^2. \tag{9}$$

The adequate choice for the shape functions $h_i(x)$ often depends on the nature of the problem and the experience of the analyst.

As an example consider the variational problem

$$I[y(x)] = \int_0^1 \left[y^2 + (y')^2\right] dx; \qquad y(0) = y(1) = 1. \tag{10}$$

Adopting the set of shape functions (8) an approximate solution is sought in the form stated by (3) as

$$\bar{y}(x) = a_1 + a_2 x + a_3 x^2, \tag{11}$$

which is rewritten for notational convenience as

$$\bar{y}(x) = \alpha + \beta x + \gamma x^2. \tag{12}$$

Considering that, according to expression (4), $\bar{y}'(x)$ can be expressed as

$$\bar{y}'(x) = \beta + 2\gamma x \tag{13}$$

and substituting (12) and (13) into (10), one gets the expression

$$\Phi(a, b, c) = \int_0^1 \left[(\alpha + \beta x + \gamma x^2)^2 + (\beta + 2\gamma x)^2\right] dx \tag{14}$$

with the restrictions resulting from the boundary conditions,

$$\begin{aligned} \bar{y}(0) = \alpha + \beta \cdot 0 + \gamma \cdot 0^2 = 1 \quad &\Rightarrow \quad \alpha \;= 1; \\ \bar{y}(1) = \alpha + \beta \cdot 1 + \gamma \cdot 1^2 = 1 \quad &\Rightarrow \quad \gamma \;= -\beta; \end{aligned} \tag{15}$$

The problem is thus reduced to finding the coefficients α, β and γ which provide an extremum of function (14) subject to the constraints (15). Substituting the constraints (15) into (14) leads to the expression

$$\Phi(\beta) = \int_0^1 \left[(1 + \beta x - \beta x^2)^2 + (\beta - 2\beta x)^2\right] \, dx, \tag{16}$$

which, after integration with respect to x, reduces to

$$\Phi(\beta) = 1 + \frac{1}{3}\beta + \frac{11}{30}\beta^2. \tag{17}$$

An extremum is found when

$$\frac{d\Phi}{d\beta} = \frac{1}{3} + \frac{11}{15}\beta = 0 \quad \Rightarrow \quad \beta = -\frac{5}{11}. \tag{18}$$

The solution to (10) is thus approximated by Ritz method as

$$\bar{y}(x) = 1 - \frac{5}{11}x + \frac{5}{11}x^2 \tag{19}$$

for the shape functions (8).

The exact solution to (10) can be shown to be

$$y(x) = \frac{\sinh(x) + \sinh(1 - x)}{\sinh(1)}. \tag{20}$$

The plots for $\bar{y}(x)$ and $y(x)$ in Figure 1 do not exhibit any appreciable difference. How powerful Ritz method actually is becomes patent on the plot of the relative error

$$\epsilon_{rel} = \frac{y(x) - \bar{y}(x)}{y(x)}$$

represented in Figure 2, where it is appreciated that the maximal relative error is found for $x = 0,5$ and amounts to $0,05\%$.

Ritz method is the mathematical foundation of the *Finite Element Method.* For the particular case of structural mechanics in static conditions the variational problem is simply the principle of stationary potential energy. By choosing the shape functions $h_i(x)$ conveniently as piece-wise, low-degree polynomials the evaluation of the integral (5) —or (16) in the example— can be expedited in such a way that a system of algebraic equations for the coefficients a_i can be automatically set up from a data file including information on the geometry of the body, the supports (aka constraints) and the applied forces. Solving this system of algebraic equations provides an approximate solution of the considered problem.

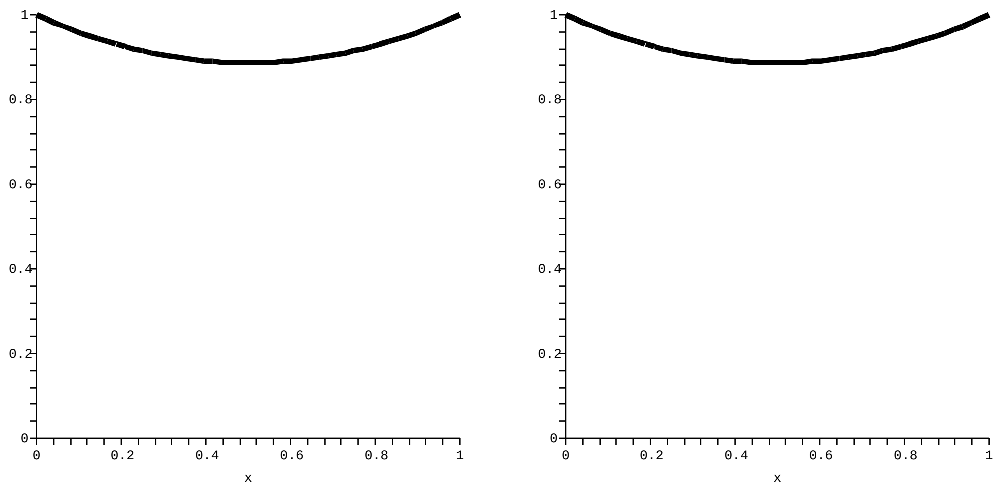

Figure 1. Plots of the approximated solution $\bar{y}(x)$ (left) and the exact solution $y(x)$ (right)

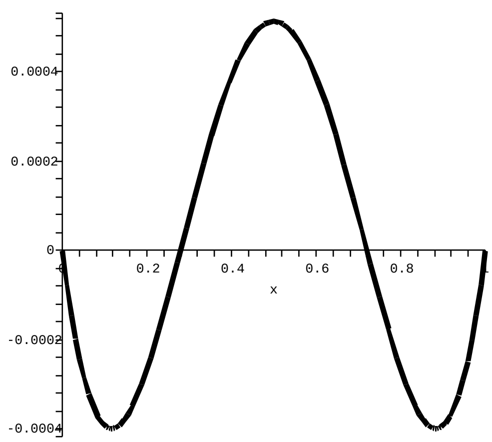

Figure 2. Plot of the relative error $\epsilon_{rel} = \dfrac{y(x) - \bar{y}(x)}{y(x)}$