Solutions to problem set 1: Quantum states and transformations

1) Separability

$$\frac{|00\rangle + i|01\rangle - |10\rangle + i|11\rangle}{2} \quad \text{is entangled}$$

$$\frac{|00\rangle + i|01\rangle - |10\rangle - i|11\rangle}{2} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$$

$$\frac{|000\rangle - |010\rangle}{\sqrt{2}} = |0\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes |0\rangle$$

$$\frac{|100\rangle + |010\rangle}{\sqrt{2}} = \left(\frac{|10\rangle + |01\rangle}{\sqrt{2}}\right) \otimes |0\rangle \quad \text{(the first two are entangled with each other)}$$

2) In the state $(|0\rangle + \frac{1-i}{\sqrt{2}}|1\rangle)/\sqrt{2}$, the weight of $|0\rangle$ and $|1\rangle$ is the same $(1/\sqrt{2})$, so the state lies along the equator, $z = 0$. Furthermore, it lies halfway between the states $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ (along the $\hat{x}$ axis) and $\frac{|0\rangle-i|1\rangle}{\sqrt{2}}$ (along the $-\hat{y}$ axis), thus $x = -y = 1/\sqrt{2}$. The spherical coordinates (θ, ϕ) are $(90°, -45°)$.

In the state $\sqrt{\frac{1}{4}}|0\rangle - \sqrt{\frac{3}{4}}|1\rangle$, $z = -1/2$ (the state lies three quarters along the way from $|0\rangle$ to $|1\rangle$). The relative phase between $|0\rangle$ and $|1\rangle$ is -1, so the state lies along the $-\hat{x}$ axis, i.e. $y = 0$. Since also $x^2 + y^2 + z^2 = 1$, we find $x = -\sqrt{3/4}$ (by coincidence, it's the same as the amplitude of $|1\rangle$). In spherical coordinates, we have $\phi = 180°$ and $\theta/2 = \arccos 1/2 = 60°$, so $\theta = 120°$.

3) This matrix is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

4) Flipping the phase of a qubit means $a|0\rangle + b|1\rangle \mapsto a|0\rangle - b|1\rangle$. This transformation is described by the 2×2 matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The 4×4 unitary transformation which flips the phase of qubit 1 if qubit 2 is in $|1\rangle$ and does nothing if qubit 2 is in $|0\rangle$ (i.e. a controlled-phase gate or CPHASE), is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(note that this is the same gate which flips the phase of qubit 2 if and only if qubit 1 is in $|1\rangle$).

This gate creates entanglement when applied, for instance, to $\frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, giving $\frac{|00\rangle+|01\rangle+|10\rangle-|11\rangle}{2}$ $(= |0\rangle\frac{|0\rangle+|1\rangle}{\sqrt{2}} + |1\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}})$.

NOTE: Notice the difference between the CPHASE gate above and the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which describes simply a phase flip of the first qubit while nothing happens to the second qubit. Indeed, we can rewrite this last matrix as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$