Solutions to problem set 2: Hamiltonian and unitary time evolution

1) We use the Taylor expansion to obtain

$$\exp(-i\alpha\sigma_x) = I + \frac{(-i\alpha\sigma_x)}{1!} + \frac{(-i\alpha\sigma_x)^2}{2!} + \frac{(-i\alpha\sigma_x)^3}{3!} + \dots$$

$$= I - i\frac{(\alpha\sigma_x)}{1!} - \frac{(\alpha\sigma_x)^2}{2!} + i\frac{(\alpha\sigma_x)^3}{3!} + \dots$$

Now we use the fact that $\sigma_x^2 = I$ and group terms containing I and terms containing σ_x:

$$= I\left[1 - \frac{\alpha^2}{2!} + \dots\right] - i\sigma_x\left[\frac{\alpha}{1!} - \frac{\alpha^3}{3!} + \dots\right]$$

$$= I\cos\alpha - i\sigma_x \sin\alpha.$$

2) **Method 1** (elegant but perhaps not easy to come up): A rotation about a general axis can be expanded as

$$U = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z).$$

For H, we see that $\theta = \pi$, and that $n_x = n_z = 1/\sqrt{2}$. So the rotation is over an angle of 180° about an axis halfway the x and the z axis.

For U, we find $\theta = \pi/2$ and $n_y = 1$. So this is a 90° rotation about the y axis.

Method 2 (less elegant but more straightforward): The rotation axis of a unitary transformation is given by the eigenvectors of the matrix describing the transformation. The eigenvalues of H are ± 1, and the eigenvectors are

$$\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$$

or, after normalization (this is not really needed to determine the point in the Bloch sphere the eigenstates corresponds to),

$$\frac{1}{\sqrt{4+2\sqrt{2}}}\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} 0.9239 \\ 0.3827 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{4-2\sqrt{2}}}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -0.3827 \\ 0.9239 \end{pmatrix}$$

Using $\psi = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, we find that these describe the states halfway between the $\hat{x}$ and $\hat{z}$ axis, and halfway between the $-\hat{x}$ and $-\hat{z}$ axis respectively. Similarly, we find that eigenvectors of U lie along $\pm\hat{y}$. The rotation angle can be derived from the eigenvalues (we won't do this here), but it's easier to find by looking at its effect on some initial states.

As a check, we see that

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}; \quad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}; \quad H\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |0\rangle;$$

This is consistent with H performing a 180° rotation about the axis halfway between the $\hat{x}$ and $\hat{z}$ axis. Analogously, we have

$$U|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}; \quad U|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}; \quad U\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |1\rangle;$$

We conclude that U performs a $90°$ rotation about the $\hat{y}$ axis. Note that H and U have the same effect when applied to $|0\rangle$ or $|1\rangle$. Only when we examine their effect on other initial states do we discover how they are different.

3) By reasoning in the Bloch sphere ($\hat{z}$ rotations change the coordinate system from $\hat{x}$ to $\hat{y}$ to $-\hat{x}$ and so forth), we find that

$$R_y(90) = R_z(90)R_x(90)R_z(-90)$$

The equality can be verified by multiplying out the three matrices on the right (I do this using simple MATLAB routines). Note that as always, the operation that is executed first is placed on the rightmost side of the product. So the first rotation is the one over $90°$ about the $-\hat{z}$ axis.

Similarly,

$$R_y(45) = R_z(90)R_x(45)R_z(-90)$$
$$R_y(180) = R_z(90)R_x(180)R_z(-90)$$