Solution set 3: Quantum circuits

1) The swap gate cannot create entanglement - it merely interchanges the state of two qubits. Obviously, single-qubit operations cannot create entanglement either. Since this set of gates cannot create entanglement, it cannot be universal.

2) In this case we start from

$$|\psi_0\rangle = (a|0\rangle + b|1\rangle)\frac{|00\rangle - |11\rangle}{\sqrt{2}}.$$

After the CNOT of the first onto the second qubit, we get

$$|\psi_1\rangle = a|0\rangle(\frac{|00\rangle - |11\rangle}{\sqrt{2}}) + b|1\rangle(\frac{|10\rangle - |01\rangle}{\sqrt{2}})$$

Following the Hadamard, this becomes

$$|\psi_2\rangle = a\frac{|0\rangle + |1\rangle}{\sqrt{2}}(\frac{|00\rangle - |11\rangle}{\sqrt{2}}) + b\frac{|0\rangle - |1\rangle}{\sqrt{2}}(\frac{|10\rangle - |01\rangle}{\sqrt{2}})$$

Rewriting gives

$$|\psi_2\rangle = \frac{1}{2}\left[|00\rangle(a|0\rangle - b|1\rangle) + |01\rangle(b|0\rangle - a|1\rangle) + |10\rangle(a|0\rangle + b|1\rangle) + |11\rangle(b|0\rangle + a|1\rangle)\right]$$

We see that, as before, the outcome of the measurement that is next performed on the first two particles, tells us what rotation to apply to the third qubit, in order to recover the original state $a|0\rangle + b|1\rangle$.

ALTERNATIVE SOLUTION: first rotate $\frac{|00\rangle - |11\rangle}{\sqrt{2}}$ into $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$, for instance by a (single-qubit) z rotation of qubit 2 over 180°. Next apply the protocol described in class.

3) The Hadamard gate on qubit 1 puts it in the superposition $(|0\rangle + |1\rangle)/\sqrt{2}$. If we next evaluate $f(x)$ and add the result (modulo 2) to the second qubit, this is done for two values of x at once, hence quantum parallellism.

Surprisingly, the two qubits never get entangled in the course of this algorithm. Certainly the Hadamards don't create entanglement (they are single-qubit gates):

$$|0\rangle|1\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}}\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Next we do the function evaluation $f(x)$ and add it to the second qubit modulo 2 (such additions are represented by the $\oplus$ symbol).

$$\mapsto \frac{1}{2}\left(|0\rangle\frac{|0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle}{\sqrt{2}} + |1\rangle\frac{|0 \oplus f(1)\rangle - |1 \oplus f(1)\rangle}{\sqrt{2}}\right)$$

Let us first look for instance at the case $f(0) = f(1) = 1$. The state is then given by

$$\frac{1}{2}\left(|0\rangle(|1\rangle - |0\rangle) + |1\rangle(|1\rangle - |0\rangle)\right) = -\frac{|0\rangle + |1\rangle}{\sqrt{2}}\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

which is separable. If we take the case $f(0) = 0$, $f(1) = 1$, the state after the function evaluation becomes

$$\frac{1}{2}\left(|0\rangle(|0\rangle - |1\rangle) + |1\rangle(|1\rangle - |0\rangle)\right) = \frac{|0\rangle - |1\rangle}{\sqrt{2}}\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

which is separable as well.

We conclude that apparently entanglement is not used in some quantum algorithms.