Solution set 4: Pure versus mixed states, density matrices

1) The density matrices for the Bell pairs are

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \Rightarrow \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \frac{|00\rangle - |11\rangle}{\sqrt{2}} \Rightarrow \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\frac{|01\rangle + |10\rangle}{\sqrt{2}} \Rightarrow \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \frac{|01\rangle - |10\rangle}{\sqrt{2}} \Rightarrow \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

When I randomly draw one pair from the bag, it can be in any of the four Bell pairs, each with probability 1/4. The density matrix describing the state of the particles is thus 1/4 the sum of the four density matrices above. This simply gives the identity matrix,

$$\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Clearly, this density matrix is not entangled. So when you're randomly given one of four orthogonal, maximally entangled states, it is not entangled!

2) For the first density matrix, we have

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \qquad \text{and} \qquad \rho_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

We can verify that

$$\rho_A \otimes \rho_B = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is equal to the density matrix describing the joint state of the two particles. Since we can properly describe the joint state as the tensor product of the state of the two individual particles, the particles are not entangled.

For the second density matrix, we find

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \rho_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now the tensor product of the two single-particle density matrices gives the 4×4 identity matrix, which is obviously different from the density matrix describing the joint state of the particles. This implies that the joint state is entangled.