Solution set 5: Non-unitary processes

1) A first scenario is the perform a projective measurement. The measurement outcome tells us onto what state the quantum system is projected, so we know the state after the measurement. A state which was mixed before the measurement will be pure after the measurement.

A second scenario, that doesn't involve a measurement, occurs for instance when a qubit is maximally entangled with another qubit. The joint system is then in a pure state, but the subsystems are in a completely mixed state (see also problem set 4). Since we know the state of the joint system, we can apply a two-qubit gate that disentangles the two qubits. The resulting joint state is separable, and each of the qubits by itself is now in a pure state.

A third scenario is relaxation to the ground state (at a temperature low compared to the energy difference between the two qubit states).

2) The second qubit is in $|0\rangle$. Let the initial state of the first qubit be

$$\rho_i^1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Method 1: The initial density matrix of the joint system is then given by

$$\rho_i = \begin{pmatrix} a & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

After the CNOT gate, we obtain

$$\rho_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 0 & d \end{pmatrix}$$

The reduced density matrix of the first qubit is now given by

$$\rho_f^1 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} .$$

An example of a process that maps ρ_i^1 onto ρ_f^1 is

$$\rho_f^1 = E_0 \rho_i^1 E_0^\dagger + E_1 \rho_i^1 E_1^\dagger$$

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In words, the state of qubit 1 is projected onto $|0\rangle$ or $|1\rangle$. Another possibility is

$$E_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In words, the phase of qubit 1 is flipped or nothing happens, with equal probabilities.

Method 2: With qubit 2 in the state $|0\rangle$, the action of the CNOT on the first qubit by itself can be written as

$$\rho_A \mapsto \sum_{k=0,1} {}_B\langle e_k|U_{cnot}\rho U_{cnot}|e_k\rangle_B = \sum_{k=0,1} {}_B\langle e_k|U_{cnot}(\rho_A \otimes |0\rangle_{B\,B}\langle 0|)U_{cnot}|e_k\rangle_B$$

$$= \sum_{k=0,1} ({}_B\langle e_k|U_{cnot}|0\rangle_B)\rho_A({}_B\langle 0|)U_{cnot}|e_k\rangle_B)$$

Using the fact that we can expand U_{cnot} as

$$U_{cnot} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

we obtain (after some algebra)

$$\rho_A \mapsto |0\rangle\langle 0|\rho_A|0\rangle\langle 0| + |1\rangle\langle 1|\rho_A|1\rangle\langle 1|.$$

Again, the process is described by the operators

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$