Solution set 6: Quantum measurement

1) For the circuit with the CNOT, we find

- After the CNOT, the joint state of qubit and meter is $a|0\rangle|0\rangle_M + b|1\rangle|1\rangle_M$.
- $P(0) = |a|^2$ and $P(1) = |b|^2$.
- The state of the qubit after projection of the meter is $|0\rangle$ for outcome 0 and $|1\rangle$ for outcome 1.
- Since the qubit is entangled with the meter, measurement of the meter indirectly measures the qubit. The POVM operators that describe this indirect qubit measurement are
$$E_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
Indeed, we can verify that $\langle\psi|E_0|\psi\rangle = (a^*\langle 0| + b^*\langle 1|)\; |0\rangle\langle 0|\; (a|0\rangle + b|1\rangle) = |a|^2 = P(0)$, and similarly $\langle\psi|E_0|\psi\rangle = |b|^2 = P(1)$. Also $E_0 + E_1 = I$, so the completeness condition is satisfied.
- The measurement operators M_m that describe the action on the qubit are
$$M_0 = |0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M_1 = |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$
We have $M_0^\dagger M_0 = E_0$, as should be the case (and furthermore $M_0 = E_0$ in this case), so $\langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|E_0|\psi\rangle = P(0)$, and similarly for outcome 1. Furthermore, the post-measurement state for outcome 0 is given by $M_0|\psi\rangle/\sqrt{P(0)} = |0\rangle\langle 0|(a|0\rangle + b|1\rangle)/|a| = a|0\rangle/|a|$, which is what we had expected (up to an irrelevant overall phase shift). Similarly, $M_1|\psi\rangle/\sqrt{P(1)} = b|1\rangle/|b|$.
- Since $M_i M_j = \delta_{ij} E_i$, this is indeed a projective measurement of the qubit.

2) For the circuit with the controlled-$\sqrt{\text{NOT}}$, we obtain

- The joint state after the controlled-$\sqrt{\text{NOT}}$ is
$$a|0\rangle|0\rangle_M + b\sqrt{\frac{-i}{2}}|1\rangle\,(|0\rangle + i|1\rangle)_M = \left(a|0\rangle + \sqrt{\frac{-i}{2}}b|1\rangle\right)|0\rangle_M + i\sqrt{\frac{-i}{2}}b|1\rangle|1\rangle_M$$
- $P(0) = |a|^2 + \frac{|b|^2}{2}$ and $P(1) = \frac{|b|^2}{2}$.

- The state of the qubit after projection of the meter is

$$\frac{a|0\rangle + \sqrt{\frac{-i}{2}}b|1\rangle}{\sqrt{|a|^2 + \frac{|b|^2}{2}}} \quad \text{for outcome 0} \quad \text{and} \quad |1\rangle \quad \text{for outcome 1.}$$

- The meter is still partly entangled with the qubit, and therefore measurement of the meter again constitutes an indirect (but now partial) measurement of the qubit. From $P(0)$ and $P(1)$, we find the POVM operators describing this indirect qubit measurement:

$$E_0 = |0\rangle\langle 0| + \frac{|1\rangle\langle 1|}{2} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \text{and} \quad E_1 = \frac{|1\rangle\langle 1|}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix}$$

  Indeed $E_0 + E_1 = I$.

- From the post-measurement state obtained above, we find for the measurement operators

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\frac{-i}{2}} \end{bmatrix} \quad \text{and} \quad M_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\frac{i}{2}} \end{bmatrix}.$$

  We verify that here too $M_0^\dagger M_0 = E_0$ and $M_1^\dagger M_1 = E_1$, so the operators give the correct probabilities for the respective measurement outcomes. Furthermore, the M_m also give the post-measurement states correctly:

$$M_0|\psi\rangle/\sqrt{P(0)} = \frac{1}{\sqrt{|a|^2 + \frac{|b|^2}{2}}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\frac{-i}{2}} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{|a|^2 + \frac{|b|^2}{2}}} \begin{pmatrix} a \\ \sqrt{\frac{-i}{2}}b \end{pmatrix} = \frac{a|0\rangle + \sqrt{\frac{-i}{2}}b|1\rangle}{\sqrt{|a|^2 + \frac{|b|^2}{2}}}$$

$$M_1|\psi\rangle/\sqrt{P(1)} = \frac{1}{\sqrt{\frac{|b|^2}{2}}} \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\frac{i}{2}} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{\frac{|b|^2}{2}}} \begin{pmatrix} 0 \\ \sqrt{\frac{i}{2}}b \end{pmatrix} = \frac{1}{\sqrt{\frac{|b|^2}{2}}} \sqrt{\frac{i}{2}}b|1\rangle = \sqrt{i}b|1\rangle/|b|$$

- This is not a projective measurement, because $M_0^\dagger M_1 \neq 0$ (the measurement operators are not orthonormal) and $M_0^\dagger M_0 \neq M_0$.