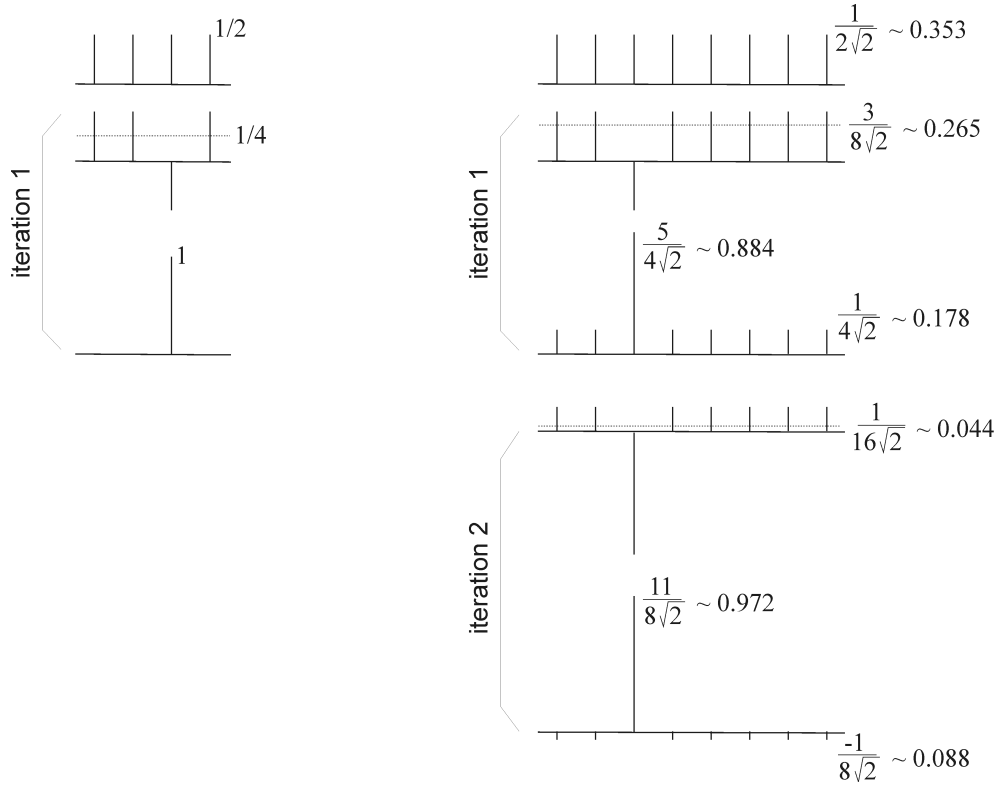
Solution set 8: Grover's algorithm

1) From the stick diagram on the left, we see that after one iteration, the marked element has amplitude 1. The probability of obtaining the marked element upon measurement is thus 1. Classically, one would randomly try one candidate solution after the other, untill the marked element is found. On average this takes $(1 + 2 + 3 + 3)/4 = 2.25$ trials. Note that if you didn't come across the marked element in the first three attempts, you automatically know the remaining element is the marked element, without a need for a fourth oracle call.



2) Following the definitions of Nielsen and Chuang, and as discussed in class, we have

$$|\alpha\rangle = \sqrt{1/8}\,(|0\rangle + |1\rangle + |3\rangle + \ldots + |7\rangle) \qquad \text{and} \qquad |\beta\rangle = \sqrt{1/8}\,|2\rangle\ .$$

The state $|\psi\rangle = \sqrt{1/8}\sum_{k=0}^{7}|k\rangle$ can then be written as

$$|\psi\rangle = \sqrt{7/8}\,|\alpha\rangle + \sqrt{1/8}\,|\beta\rangle\ .$$

It follows that the angle between $|\psi\rangle$ and $|\alpha\rangle$ is

$$\theta/2 = \arcsin\sqrt{1/8} = 20.705^\circ\ .$$

After the first Grover iteration, we obtain a state an angle $3\theta/62.114^\circ$ away from $|\alpha\rangle$. After the second iteration, the state ends up at an angle $5\theta/103.524^\circ$ from $|\alpha\rangle$. The state vector has

now rotated past $|\beta\rangle$. If we continue to rotate, we will rotate farther away from the desired final state, $|\beta\rangle$, so we shouldn't do a third iteration. Furthermore, we see that the state is closer to $|\beta\rangle$ after the second iteration than it was after the first iteration. The optimal number of iterations is thus 2.

At this point, the probability of success is given by the overlap squared between the state we obtained and $|\beta\rangle$, which is $|\sin 103.524|^2 = 94.5\%$.

Classically, one would need on average $(1+2+3+4+5+6+7+7)/8 = 4.375$ trials.

For comparison (this was not asked in the homework), I also show the stick diagrams for this case. After the first iteration, the probability of success is $25/32 = 78.1\%$. After the second iteration, the probability of success has increased to $121/128 = 94.5\%$ (note that the amplitudes satisfy the normalization condition at all times). Since the amplitude of the non-marked elements is now negative, all amplitudes will be negative after the next oracle call (after the next flip of the third element). If all amplitudes are negative, the mean is also negative, and inversion about the mean of the third element will reduce the amplitude of the third element. Therefore, the optimal number of executions of the iteration step is 2.

3) Recognizing the solution(s) is not the same as knowing the solution(s) in advance. For instance, given a Boolean expression, it is easy to verify whether the expression returns 1 if you put in a given bit string (this is recognizing a solution). However, that doesn't imply you know in advance for which bit strings the expression will return 1. The latter question may be difficult to answer.

A nice answer from one of the solutions people turned in: it may take a long time to find your keys, but once you come accross them, you recognize they're yours .... Furthermore, being able to recognize your keys when you see them, doesn't mean you never have to look for them!