Solution set 9: Quantum error correction

1) A 60° single-qubit rotation about $\hat{x}$ is represented by the operator

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{i}{2} \\ -\frac{i}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} ,$$

i.e. it maps $|0\rangle$ onto $\frac{\sqrt{3}}{2}|0\rangle - \frac{i}{2}|1\rangle$ and it maps $|1\rangle$ onto $-\frac{i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. If we apply this operation to the third qubit in the encoded state $|\psi_C\rangle = a|000\rangle + b|111\rangle$, we obtain

$$\begin{aligned} & a\left(\frac{\sqrt{3}}{2}|000\rangle - \frac{i}{2}|001\rangle\right) + b\left(-\frac{i}{2}|110\rangle + \frac{\sqrt{3}}{2}|111\rangle\right) \\ &= \frac{\sqrt{3}}{2}\left(a|000\rangle + b|111\rangle\right) - \frac{i}{2}\left(a|001\rangle + b|110\rangle\right) \ . \end{aligned}$$

We now entangle the three qubits with two additional qubits (ancilla), so that the ancilla's contain Z_1Z_2 and Z_2Z_3. This gives

$$\frac{\sqrt{3}}{2}\left(a|000\rangle + b|111\rangle\right)|00\rangle - \frac{i}{2}\left(a|001\rangle + b|110\rangle\right)|01\rangle \ .$$

We then measure the two ancilla qubits. With probability 3/4, this measurement gives 00. When that happens, the state of the first three qubits has collapsed to $a|000\rangle + b|111\rangle$, and we are done. With probability 1/4, the measurement gives 01, and the first three qubits will have collapsed to $a|001\rangle + b|110\rangle$. From the syndrome outcome 01, we know that we must flip the third qubit to recover $|\psi_C\rangle$.

So in effect, the syndrome measurement turns the 60° $\hat{x}$ rotation into either no error, or a bit flip error (the error is digitized). The syndrome outcome tells us what correction operation is needed to recover $|\psi_C\rangle$.

The 60° rotation about $\hat{x}$ can be described by a single operator E_1, with $E_1 = \cos(\pi/6)I - i\sin(\pi/6)X$. This expression can also be interpreted as a discretization of the error process, where with probability $\cos^2(\pi/6) = 3/4$, no error occurs, and with probability 1/4 an X error occurs. Note though that it is the syndrome measurement that forces the qubit to "'choose"' between no error or a complete bit flip error.

2) We saw in class a circuit that implements Z_1Z_2: it consists of a CNOT of qubit 1 onto an ancilla initialized in $|0\rangle$, followed by a similar CNOT of qubit 2 onto the same ancilla, and then measurement of the ancilla. The measurement will give outcome 0 if qubits 1 and 2 are equal, and if will give outcome 1 if they are opposite.

Because $Z = HXH$, we can convert the circuit for Z_1Z_2 into a circuit that implements the X_1X_2 syndrome measurement by having Hadamard gates on qubits 1 and 2 before and after the CNOT's.

3) The 9-qubit Shor code is given by

$$|0\rangle_L = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle_L = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

First we note that

$$X_1 X_2 X_3(|000\rangle + |111\rangle) = (|111\rangle + |000\rangle) = (+1)(|000\rangle + |111\rangle)$$

$$X_1 X_2 X_3(|000\rangle - |111\rangle) = (|111\rangle - |000\rangle) = (-1)(|000\rangle - |111\rangle)$$

So measurement of $X_1X_2X_3$ returns $+1$ for a positive phase and -1 for a negative phase. By extension, measurement of $X_1X_2X_3X_4X_5X_6$ will return $+1$ if the phases of the first and second block are the same, and -1 if the phases are opposite.