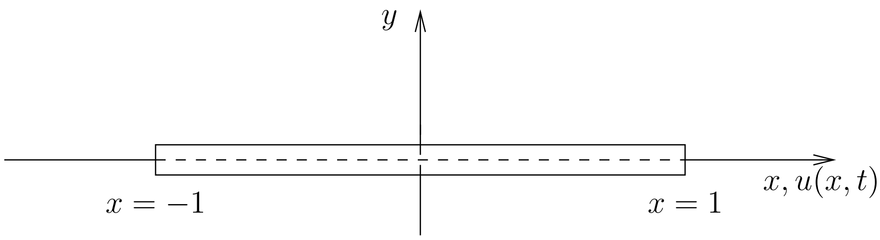
Dynamics and Stability AE3-914

Sample problem—Week 7

Ritz method for strain waves in a bar

Statement



Consider the uniform bar with volumetric density $\rho[\text{kg/m}^3]$, cross-sectional area $A[\text{m}^2]$, elasticity modulus $E[\text{N/m}^2]$ and length $2\,\text{m}$ represented in the figure. Making use of Ritz method with three degrees-of-freedom, the displacement field $u(x,t)$ is approximated as

$$\bar{u}(x,t) = \sum_{i=1}^{3} a_i(t)\, h_i(x) \tag{1}$$

with the shape functions

$$h_1(x) = 1; \quad h_2(x) = x; \quad h_3(x) = x^2. \tag{2}$$

Find the equations of motion for the degrees-of-freedom $\{a_1(t), a_2(t), a_3(t)\}$.

Action integral

For a differential element of the bar one can write

$$dT = \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dm = \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 dx \tag{3}$$

for the kinetic energy and

$$dV = \frac{1}{2} E A \varepsilon^2 dx = \frac{1}{2} E A \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{4}$$

for the potential energy. The total kinetic energy is then

$$T = \int_{-1}^{1} \frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 dx \tag{5}$$

and the total potential energy is

$$V = \int_{-1}^{1} \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{6}$$

The action integral can then be written as

$$\begin{aligned} I[u(x,t)] &= \int_{t_1}^{t_2} (T - V) dt \\ &= \int_{t_1}^{t_2} \int_{-1}^{1} \left[\frac{1}{2}\rho A \left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2} EA \left(\frac{\partial u}{\partial x}\right)^2 \right] dx dt \end{aligned} \tag{7}$$

Ritz approximation

An approximate solution is sought by substituting (1) into the action (7). It is convenient to keep the generic notation $h_i(x)$ as far as possible rather than substituting the actual provided expressions (2) for the shape functions. The time and space derivatives of $\bar{u}$ are

$$\frac{\partial \bar{u}}{\partial t} = \sum_{i=1}^{3} \dot{a}_i(t) h_i(x) \quad \text{and} \quad \frac{\partial \bar{u}}{\partial x} = \sum_{i=1}^{3} a_i(t) h_i'(x), \tag{8}$$

respectively. The action integral is then expressed as

$$I[\bar{u}(x,t)] = \int_{t_1}^{t_2} \int_{-1}^{1} \left[\frac{1}{2}\rho A \left(\sum_{i=1}^{3} \dot{a}_i(t) h_i(x) \right)^2 - \frac{1}{2} EA \left(\sum_{i=1}^{3} a_i(t) h_i'(x) \right)^2 \right] dx dt, \tag{9}$$

which is elaborated —still in terms of generic $h_i(x)$— as

$$\begin{aligned} I[\bar{u}(x,t)] = \int_{t_1}^{t_2} \int_{-1}^{1} \Bigg\{ & \frac{1}{2}\rho A \left[\dot{a}_1^2 h_1^2 + \dot{a}_2^2 h_2^2 + \dot{a}_3^2 h_3^2 \right. \\ & \qquad \left. + 2\dot{a}_1 \dot{a}_2 h_1 h_2 + 2\dot{a}_1 \dot{a}_3 h_1 h_3 + 2\dot{a}_2 \dot{a}_3 h_2 h_3 \right] \\ & - \frac{1}{2} EA \left[a_1^2 {h_1'}^2 + a_2^2 {h_2'}^2 + a_3^2 {h_3'}^2 \right. \\ & \qquad \left. + 2a_1 a_2 h_1' h_2' + 2a_1 a_3 h_1' h_3' + 2a_2 a_3 h_2' h_3' \right] \Bigg\} dx dt \end{aligned} \tag{10}$$

where the explicit dependence of a_i on t and h_i on x is not longer emphasised for notational simplicity. Equation (10) contains two quadratic forms —comparable to those found when deriving the inertia tensor— and can therefore be rewritten as

$$\begin{aligned} I[\bar{u}] = \int_{t_1}^{t_2} \int_{-1}^{1} \Bigg\{ & \frac{1}{2}\rho A \begin{pmatrix} \dot{a}_1 & \dot{a}_2 & \dot{a}_3 \end{pmatrix} \begin{bmatrix} h_1^2 & h_1 h_2 & h_1 h_3 \\ h_1 h_2 & h_2^2 & h_2 h_3 \\ h_1 h_3 & h_2 h_3 & h_3^2 \end{bmatrix} \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \end{pmatrix} \\ & - \frac{1}{2} EA \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{bmatrix} {h_1'}^2 & h_1' h_2' & h_1' h_3' \\ h_1' h_2' & {h_2'}^2 & h_2' h_3' \\ h_1' h_3' & h_2' h_3' & {h_3'}^2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Bigg\} dx dt \end{aligned} \tag{11}$$

Considering the given shape functions (2)

$$h_1 = 1; \quad h_2 = x; \quad h_3 = x^2 \tag{12}$$

and their derivatives

$$h'_1 = 0; \quad h'_2 = 1; \quad h'_3 = 2x, \tag{13}$$

expression (11) can be reworked as

$$\begin{aligned} I[\bar{u}] = \int_{t_1}^{t_2} \Bigg\{ & \frac{1}{2}\rho A \begin{pmatrix} \dot{a}_1 & \dot{a}_2 & \dot{a}_3 \end{pmatrix} \int_{-1}^{1} \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix} dx \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \end{pmatrix} \\ & - \frac{1}{2}EA \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \int_{-1}^{1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2x \\ 0 & 2x & 4x^2 \end{bmatrix} dx \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Bigg\} dt \end{aligned} \tag{14}$$

for which the integrals with respect to x are carried out to obtain an action functional depending on the degrees-of-freedom a_i only,

$$\begin{aligned} I[a_1, a_2, a_3] = \int_{t_1}^{t_2} \Bigg\{ & \frac{1}{2}\rho A \begin{pmatrix} \dot{a}_1 & \dot{a}_2 & \dot{a}_3 \end{pmatrix} \begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{bmatrix} \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \end{pmatrix} \\ & - \frac{1}{2}EA \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Bigg\} dt \end{aligned} \tag{15}$$

which, after elaboration of the quadratic forms, renders

$$I[a_1, a_2, a_3] = \int_{t_1}^{t_2} \left[\frac{1}{2}\rho A \left(2\dot{a}_1^2 + \frac{2}{3}\dot{a}_2^2 + \frac{2}{5}\dot{a}_3^2 + \frac{4}{3}\dot{a}_1\dot{a}_3 \right) - \frac{1}{2}EA \left(2a_2^2 + \frac{8}{3}a_3^2 \right) \right] dt. \tag{16}$$

This equation can be identified as an action integral on a Lagrangian depending on three degrees-of-freedom (aka generalised coordinates), i.e.,

$$I[a_1, a_2, a_3] = \int_{t_1}^{t_2} L(a_1, a_2, a_3, \dot{a}_1, \dot{a}_2, \dot{a}_3)\, dt \tag{17}$$

Equations of motion

The equations of motion for the degrees-of-freedom a_i can be obtained from the Lagrangian in equation (16), through the standard procedure

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}_i}\right) - \frac{\partial L}{\partial a_i} = 0 \quad i = 1, 2, 3, \tag{18}$$

which is immediately elaborated as

$$
\begin{aligned}
0 &= 2\ddot{a}_1 + \frac{2}{3}\ddot{a}_3 \\
0 &= \rho\frac{2}{3}\ddot{a}_2 + E2a_2 \\
0 &= \rho\left(\frac{2}{3}\ddot{a}_1 + \frac{2}{5}\ddot{a}_3\right) + E\frac{8}{3}a_3
\end{aligned}
\tag{19}
$$

and can be written in matrix form as

$$
\rho\begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{bmatrix}\begin{pmatrix} \ddot{a}_1 \\ \ddot{a}_2 \\ \ddot{a}_3 \end{pmatrix} + E\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\tag{20}
$$

where a three degrees-of-freedom free-vibration problem is recognised.

Remarks

If static conditions are considered then the equations of motion (18) reduce to

$$
\frac{\partial L}{\partial a_i} = 0 \quad i = 1, 2, 3,
\tag{21}
$$

in which a search for a extremum of function L is recognised as in the basic procedure of Ritz method explained in the handout. It is left to the reader to explain why the matrix resulting from system (21), which is found in the right term of (20), is singular and why, in consequence, there is no solution for a_i in static conditions.