Dynamics & Stability

AE3-914

$$
\begin{aligned}
T &= \frac{1}{2}\int (\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}) \cdot (\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r})\, dm \\
&= \frac{1}{2} m v_o^2 + \mathbf{v}_o \cdot \boldsymbol{\omega} \times \int \mathbf{r}\, dm \\
&\qquad\qquad + \frac{1}{2}\int (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})\, dm
\end{aligned}
$$

$$\boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

$$= (\omega_y z - \omega_z y)\mathbf{i} + (\omega_z x - \omega_x z)\mathbf{j} + (\omega_x y - \omega_y x)\mathbf{k}$$

$$(\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\mathbf{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

is the inertia tensor

Inertia tensor

$\mathbf{I}_O$ w.r.t. the fixed point O

$\mathbf{I}_G$ w.r.t. the mass centre G

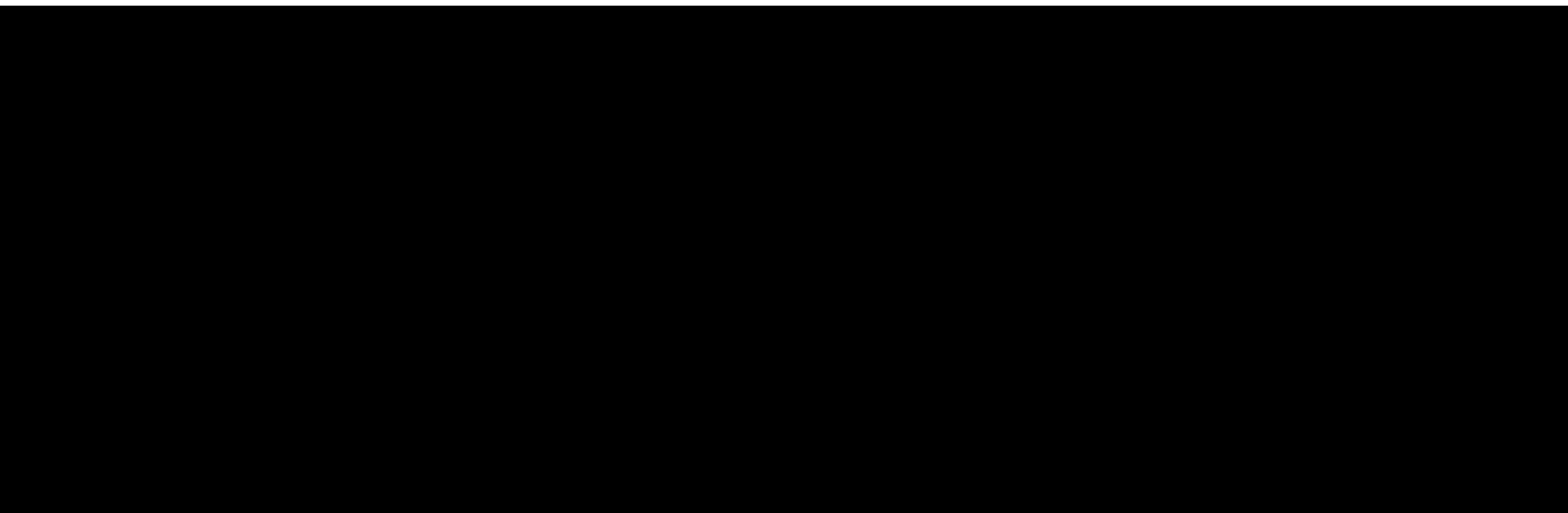
Fixed point rotation: $T = \frac{1}{2}\boldsymbol{\omega}^T \mathbf{I}_O \boldsymbol{\omega}$

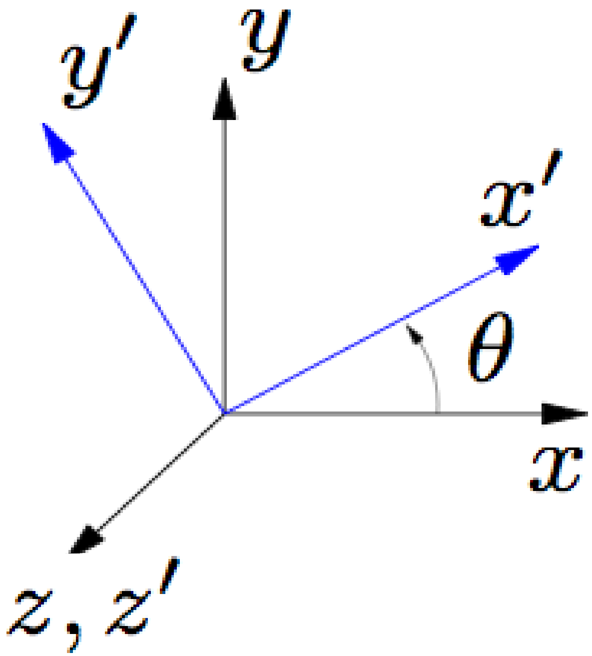
General motion: $T = \frac{1}{2} m v_G^2 + \frac{1}{2}\boldsymbol{\omega}^T \mathbf{I}_G \boldsymbol{\omega}$

I depends on:

(a) Origin of coordinates
(b) Direction of axes

The coordinate system can be rotated





$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = [\mathrm{T}] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

$$\mathrm{v}' = \mathrm{Tv}$$

$$\left[\mathrm{I}'\right] = [\mathrm{T}][\mathrm{I}][\mathrm{T}]^{T}$$

I is symmetric:

(a) **I** can be diagonalised (eigenvalues)
(b) Eigenvectors (principal directions) are orthogonal

which can be reached through a coordinate transformation

Principal moments of inertia:

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Principal axes $\rightarrow$ Diagonal form

Symmetry axes are principal axes

Planar body in *xy*-plane:

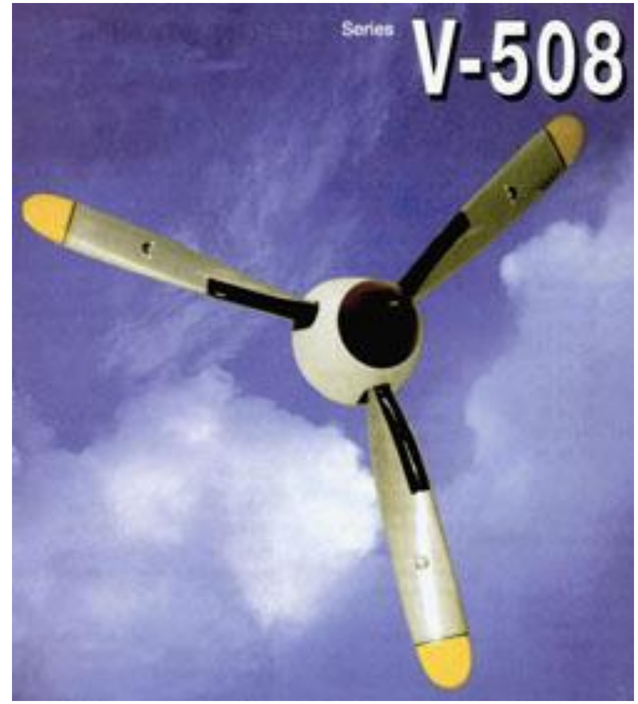
$$I_{zz} = I_{xx} + I_{yy}; \quad I_{xz} = I_{yz} = 0$$

Body with at least **three** symmetry axes in *xy*-plane:

$$I_{xx} = I_{yy}; \quad I_{xy} = 0$$

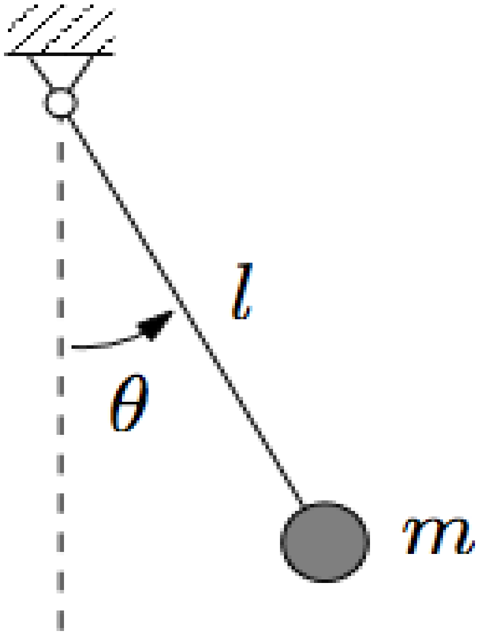
for any frame rotation about *z*-axis

Find the inertia tensor given the moment of inertia of one blade I_b about the shaft. Model the blade as a bar.



Virtual work

Please check the book!



$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Generalised coordinates

Particle in 2D: 2 degrees-of-freedom (dof)
Body in 2D: 3 dof

Particle in 3D: 3 dof
Body in 3D: 6 dof

Two bodies in 2D: 6 dof

N particles in 3D: 3N dof

and so on...

a) 1 dof

b) 2 dof

c) 3 dof

a) 1 dof

b) 2 dof

c) 3 dof

Constraints

$$f_i(q_1, \ldots, q_n) = 0$$

or

$$f_i(q_1, \ldots, q_n; t) = 0$$

$$n_{dof} = n_{coord} - n_{cons}$$

Purpose:

Find equations of motion and energy expressions in generalised coordinates

$$“\ \mathbf{F} = m\mathbf{a}\ “$$

Kinetic energy

A particle moving in space:

$$x_i = x_i(q_1, q_2, q_3, t)$$

$$\dot{x}_i = \sum_{j=1}^{3} \left(\frac{\partial x_i}{\partial q_j} \dot{q}_j \right) + \frac{\partial x_i}{\partial t}$$

$$
\begin{aligned}
T &= \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 \\
&= \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} \alpha_{jk} \dot{q}_j \dot{q}_k + \sum_{j=1}^{3} \beta_j \dot{q}_j + \gamma
\end{aligned}
$$

$$\alpha_{jk} = \alpha_{jk}(\mathbf{q}, t);\ \ \beta_j = \beta_j(\mathbf{q}, t);\ \ \gamma = \gamma(\mathbf{q}, t)$$

$$T = T(\mathbf{q}, \dot{\mathbf{q}}, t)$$

$$T = T_2 + T_1 + T_0$$

Generalised momenta

$$p_i = \frac{\partial T}{\partial \dot{q}_i}$$

Generalised forces

For a system of *n* particles the virtual work is:

$$\delta W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i$$

$$\mathbf{F}_i = F_{ix}\mathbf{i} + F_{iy}\mathbf{j} + F_{iz}\mathbf{k}$$

$$\delta x_i = \sum_{j=1}^{ndof} \frac{\partial x_i}{\partial q_j}\delta q_j \quad \delta y_i = \sum_{j=1}^{ndof} \frac{\partial y_i}{\partial q_j}\delta q_j \quad \delta x_i = \sum_{j=1}^{ndof} \frac{\partial z_i}{\partial q_j}\delta q_j$$

$$
\begin{aligned}
\delta W &= \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i \\
&= \sum_{j=1}^{ndof} \left[\sum_{i=1}^{n} \left(F_{ix} \frac{\partial x_i}{\partial q_j} + F_{iy} \frac{\partial y_i}{\partial q_j} + F_{iz} \frac{\partial z_i}{\partial q_j} \right) \right] \delta q_j \\
&= \sum_{j=1}^{ndof} Q_j \delta q_j = \mathbf{Q} \cdot \delta \mathbf{q}
\end{aligned}
$$

The **generalised force** then becomes:

$$Q_j = \sum_{i=1}^{n} \left(F_{ix} \frac{\partial x_i}{\partial q_j} + F_{iy} \frac{\partial y_i}{\partial q_j} + F_{iz} \frac{\partial z_i}{\partial q_j} \right)$$