Dynamics & Stability

AE3-914

Generalised coordinates:

$$
\begin{aligned}
q_i &= q_i(x_1, x_2, \ldots, x_n, t) \\
x_i &= x_i(q_1, q_2, \ldots, q_{ndof}, t)
\end{aligned}
$$

Kinetic energy

A particle moving in space:

$$x_i = x_i(q_1, q_2, q_3, t)$$

$$\dot{x}_i = \sum_{j=1}^{3} \left(\frac{\partial x_i}{\partial q_j} \dot{q}_j \right) + \frac{\partial x_i}{\partial t}$$

$$
\begin{aligned}
T &= \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 \\
&= \frac{1}{2} \sum_{j=1}^{3} \sum_{k=1}^{3} \alpha_{jk} \dot{q}_j \dot{q}_k + \sum_{j=1}^{3} \beta_j \dot{q}_j + \gamma
\end{aligned}
$$

$$\alpha_{jk} = \alpha_{jk}(\mathbf{q}, t); \;\; \beta_j = \beta_j(\mathbf{q}, t); \;\; \gamma = \gamma(\mathbf{q}, t)$$

$$T = T(\mathbf{q}, \dot{\mathbf{q}}, t)$$

$$T = T_2 + T_1 + T_0$$

Generalised momenta

$$p_i = \frac{\partial T}{\partial \dot{q}_i}$$

Generalised forces

For a system of *n* particles the virtual work is:

$$\delta W = \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i$$

$$\mathbf{F}_i = F_{ix}\mathbf{i} + F_{iy}\mathbf{j} + F_{iz}\mathbf{k}$$

$$\delta x_i = \sum_{j=1}^{ndof} \frac{\partial x_i}{\partial q_j} \delta q_j \qquad \delta y_i = \sum_{j=1}^{ndof} \frac{\partial y_i}{\partial q_j} \delta q_j \qquad \delta z_i = \sum_{j=1}^{ndof} \frac{\partial z_i}{\partial q_j} \delta q_j$$

$$
\begin{aligned}
\delta W &= \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i \\
&= \sum_{j=1}^{ndof} \left[\sum_{i=1}^{n} \left(F_{ix} \frac{\partial x_i}{\partial q_j} + F_{iy} \frac{\partial y_i}{\partial q_j} + F_{iz} \frac{\partial z_i}{\partial q_j} \right) \right] \delta q_j \\
&= \sum_{j=1}^{ndof} Q_j \delta q_j = \mathbf{Q} \cdot \delta \mathbf{q}
\end{aligned}
$$

The **generalised force** then becomes:

$$Q_j = \sum_{i=1}^{n} \left(F_{ix} \frac{\partial x_i}{\partial q_j} + F_{iy} \frac{\partial y_i}{\partial q_j} + F_{iz} \frac{\partial z_i}{\partial q_j} \right)$$

Virtual Work in terms of generalised forces:

$$\delta W = \sum_{i=1}^{ndof} Q_i \, \delta q_i$$

Conservative forces:

$$V = V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \qquad \mathbf{F}_i = -\nabla_i V$$

$$\nabla_i = \frac{\partial}{\partial x_i}\mathbf{i} + \frac{\partial}{\partial y_i}\mathbf{j} + \frac{\partial}{\partial z_i}\mathbf{k}$$

$$\begin{aligned}\delta W &= \sum_{i=1}^{n} \mathbf{F}_i \cdot \delta \mathbf{r}_i \\ &= -\sum_{i=1}^{n} \left(\frac{\partial V}{\partial x_i} \delta x_i + \frac{\partial V}{\partial y_i} \delta y_i + \frac{\partial V}{\partial z_i} \delta z_i \right) \\ &= -\delta V\end{aligned}$$

Expressing *V* in generalised coordinates:

$$V = V(q_1, q_2, \ldots, q_{ndof})$$

$$\begin{aligned} \delta W &= -\delta V \\ &= -\sum_{j=1}^{ndof} \frac{\partial V}{\partial q_j} \delta q_j \end{aligned}$$

Then, for conservative systems one has:

$$Q_j = -\frac{\partial V}{\partial q_j}$$

Lagrangian dynamics

Second Newton's law:

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt}$$

where

$$\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$$

Kinetic energy:

$$T = \frac{1}{2} \sum_{i=1}^{n} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right)$$

$$
\begin{aligned}
p_k &= \frac{\partial T}{\partial \dot{q}_k} \\
&= \sum_{i=1}^{n} m_i \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} + \dot{y}_i \frac{\partial \dot{y}_i}{\partial \dot{q}_k} + \dot{z}_i \frac{\partial \dot{z}_i}{\partial \dot{q}_k} \right)
\end{aligned}
$$

Chain rule (also for *y* and *z*):

$$\dot{x}_i = \sum_{j=1}^{ndof} \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}$$

from which it follows that:

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k}$$

Substituting this relation in the linear momentum:

$$p_k = \sum_{i=1}^{n} m_i \left(\dot{x}_i \frac{\partial x_i}{\partial q_k} + \dot{y}_i \frac{\partial y_i}{\partial q_k} + \dot{z}_i \frac{\partial z_i}{\partial q_k} \right)$$

Taking the time derivative of the linear momentum:

$$\frac{dp_k}{dt} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_{i=1}^{n} m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_k} + \ddot{y}_i \frac{\partial y_i}{\partial q_k} + \ddot{z}_i \frac{\partial z_i}{\partial q_k}\right)$$

$$+ \sum_{i=1}^{n} m_i \left[\dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) + \dot{y}_i \frac{d}{dt}\left(\frac{\partial y_i}{\partial q_k}\right) + \dot{z}_i \frac{d}{dt}\left(\frac{\partial z_i}{\partial q_k}\right)\right]$$

The first summation can be written as:

$$\sum_{i=1}^{n} m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_k} + \ddot{y}_i \frac{\partial y_i}{\partial q_k} + \ddot{z}_i \frac{\partial z_i}{\partial q_k} \right)$$

$$= \sum_{i=1}^{n} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k} \right)$$

$$= Q_k$$

For the second summation we note that:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) &= \sum_{j=1}^{ndof} \frac{\partial^2 x_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 x_i}{\partial t \partial q_k} \\
&= \frac{\partial}{\partial q_k}\left[\sum_{j=1}^{ndof} \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}\right] \\
&= \frac{\partial \dot{x}_i}{\partial q_k}
\end{aligned}$$

$$\dot{p}_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k + \sum_{i=1}^{n} m_i \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k} + \dot{y}_i \frac{\partial \dot{y}_i}{\partial q_k} + \dot{z}_i \frac{\partial \dot{z}_i}{\partial q_k} \right)$$

$$= Q_k + \frac{\partial}{\partial q_k} \left[\frac{1}{2} \sum_{i=1}^{n} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right) \right]$$

$$= Q_k + \frac{\partial T}{\partial q_k}$$

Rewriting this relation gives the

Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \quad k = 1, \ldots, ndof$$

For conservative forces one has:

$$Q_j = -\frac{\partial V}{\partial q_j}$$

Lagrangian function

$$L = T - V$$

Lagrange equations of motion (conservative systems)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad k = 1, \ldots, ndof$$

General case (non-conservative systems)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q_k^{nc} \quad k = 1, \ldots, ndof$$