# Dynamics & Stability

## AE3-914

If we have a generalised coordinate *q* of a Lagrangian system such that

$$L = L(\dot{q}) \qquad \text{but} \qquad L \neq L(q)$$

then *q* is an **ignorable coordinate**

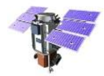
If $q$ is an ignorable coordinate

$$\frac{\partial L}{\partial q} = 0$$ and consequently

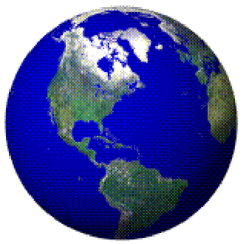
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad ==> \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\frac{\partial L}{\partial \dot{q}} = C_q$$

the generalised momentum associated with $q$ is an **integral of motion**



# Satellite system



$$V = -GM_e \frac{m}{r}$$

Consider a system of n dofs with the last m being ignorable, then

$$\frac{\partial L}{\partial \dot{q}_i} = C_i \quad i = n - m + 1, \ldots n$$

The **Routhian function** is defined as

$$R = \sum_{i=n-m+1}^{n} C_i \dot{q}_i - L$$

and is equivalent to the Lagrangian, without the ignorable coordinates

If $i \leq n - m$

$$\frac{\partial R}{\partial \dot{q}_i} = -\frac{\partial L}{\partial \dot{q}_i}$$

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If $i > n - m$

$$\frac{\partial R}{\partial \dot{q}_i} = 0$$

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The equations of motion for the non-ignorable coordinates are

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_k}\right)-\frac{\partial R}{\partial q_k}=0 \qquad k=1,\dots\ n-m$$

and the Jacobi energy integral (with the summation relating to the *non-ignorable coordinates*) is

$$h = R - \sum_{k=1}^{n-m} \dot{q}_k \frac{\partial R}{\partial \dot{q}_k}$$

# Steady motion

For the *non-ignorable coordinates*:

$$\dot{q}_k = 0 \quad \dot{p}_k = 0 \quad k = 1, \ldots n - m$$

$$\frac{\partial R}{\partial q_k} = 0 \quad k = 1, \ldots n - m$$

# Dissipative forces

$$F_x = -c_x \dot{x} \quad F_y = -c_y \dot{y} \quad F_z = -c_z \dot{z}$$

$$
\begin{aligned}
\delta W &= \mathbf{F} \cdot \delta \mathbf{r} \\
&= -(c_x \dot{x} \delta x + c_y \dot{y} \delta y + c_z \dot{z} \delta z) \\
&= -\left( c_x \dot{x} \frac{\partial x}{\partial q} + c_y \dot{y} \frac{\partial y}{\partial q} + c_z \dot{z} \frac{\partial z}{\partial q} \right) \delta q
\end{aligned}
$$

$$x_i = \hat{x}_i(q_1, \ldots q_n, t)$$

$$\dot{x}_i = \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t} \Rightarrow \frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k}$$

same for $y$ and $z$

$$
\begin{aligned}
\delta W &= -\left( c_x \dot{x} \frac{\partial \dot{x}}{\partial \dot{q}} + c_y \dot{y} \frac{\partial \dot{y}}{\partial \dot{q}} + c_z \dot{z} \frac{\partial \dot{z}}{\partial \dot{q}} \right) \delta q \\
&= -\left( \frac{1}{2} \frac{\partial}{\partial \dot{q}} (c_x \dot{x}^2 + c_y \dot{y}^2 + c_z \dot{z}^2) \right) \delta q \\
&= Q \delta q
\end{aligned}
$$

$$Q = -\frac{1}{2}\frac{\partial}{\partial \dot{q}}(c_x\dot{x}^2 + c_y\dot{y}^2 + c_z\dot{z}^2) = -\frac{\partial D}{\partial \dot{q}}$$

$$D = \frac{1}{2}(c_x\dot{x}^2 + c_y\dot{y}^2 + c_z\dot{z}^2)$$

is the **Rayleigh dissipation function**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = Q_k^* \quad k = 1, \ldots ndof$$

check the book for examples!

# Lagrange multipliers

# Analysis of contact between tyre and ground

Consider a system described by n variables and m constraints:

$$\{q_1, \ldots, q_n\}$$

$$f_j(q_1, \ldots, q_n) = 0 \quad j = 1, \ldots m$$

$$ndof = n - m$$

Constraints introduce reaction forces.

In $\mathbb{R}^3$ we have $f(x, y, z) = 0$ which provides a reaction in the direction $\perp$ to $f = 0$

$$\mathbf{R} = \lambda \nabla f = \lambda \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \qquad \delta W = \mathbf{R} \cdot \delta \mathbf{r} = 0$$

# In generalised coordinates we get the same

$$f(q_1, \ldots, q_n) = 0$$

$$\delta f = \frac{\partial f}{\partial q_1}\delta q_1 + \ldots + \frac{\partial f}{\partial q_n}\delta q_n = 0$$

$$\delta f = \mathbf{A} \cdot \delta \mathbf{q} = 0$$

$$\delta W = \mathbf{Q}^R \cdot \delta \mathbf{q} = 0$$

As a consequence $\mathbf{Q}^R = \lambda \mathbf{A}$ and

$$Q_i^R = \lambda A_i = \lambda \frac{\partial f}{\partial q_i}$$

$\lambda = \lambda(t)$ is a **Lagrange multiplier**

(extra unknown)

# Equations of motion with constraints (formulated for *n* coordinates (instead of *ndof*))

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_k} \quad k = 1, \ldots, n$$

$$f_j(q_1, \ldots, q_n) = 0 \quad j = 1, \ldots m$$

When should you use Lagrange multipliers?

(1) When the identification of degrees of freedom is difficult.

(2) When reaction or connection forces need to be evaluated.