Dynamics & Stability

AE3-914

Lagrange multipliers (review)

Consider a system described by n variables and m constraints:

$$\{q_1, \ldots, q_n\}$$

$$f_j(q_1, \ldots, q_n) = 0 \quad j = 1, \ldots m$$

$$ndof = n - m$$

In generalised coordinates constraints can have the form:

$$f(q_1, \ldots, q_n) = 0$$: *Holonomic constraint*

$$\delta f = \frac{\partial f}{\partial q_1}\delta q_1 + \ldots + \frac{\partial f}{\partial q_n}\delta q_n = 0$$

$$\delta f = \mathbf{A} \cdot \delta \mathbf{q} = 0$$: *Nonholonomic constraint*

$$\delta W = \mathbf{Q}^R \cdot \delta \mathbf{q} = 0$$

As a consequence $\mathbf{Q}^R = \lambda \mathbf{A}$ and

$$Q_i^R = \lambda A_i = \lambda \frac{\partial f}{\partial q_i}$$

Equations of motion with constraints (formulated for *n* coordinates (instead of *ndof*)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_k} \quad k = 1, \ldots, n$$

$$f_j(q_1, \ldots, q_n) = 0 \quad j = 1, \ldots m$$

Augmented Lagrangian $$L^* = L + \sum_{j=1}^{m} \lambda_j f_j$$

for the variables $\{q_1, \ldots, q_n\}$ and $\{\lambda_1, \ldots, \lambda_m\}$

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \dot{q}_k}\right) - \frac{\partial L^*}{\partial q_k} = 0 \qquad k = 1, \ldots, n$$

When should you use Lagrange multipliers?

(1) When the identification of degrees of freedom is difficult.

(2) When reaction or connection forces need to be evaluated.

Stability issues

Dynamical systems:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}(\mathbf{x}, t) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned}$$

with $\mathbf{x} \in \mathbb{R}^n$

Equilibrium point:

$$\mathbf{F}(\mathbf{x}^*, t) = \mathbf{0}$$

Is $\mathbf{x}^*$ stable or unstable?

$\mathbf{x}^*$ is stable if and only if

$$\forall \varepsilon > 0 \; \exists \delta(\varepsilon) > 0 \; \Big| \; \|\mathbf{x}_0 - \mathbf{x}^*\| < \delta \Rightarrow \|\mathbf{x}(t) - \mathbf{x}^*\| < \varepsilon$$

If

$$\exists \varepsilon > 0 \;\Big|\; \forall \delta > 0, \|\mathbf{x}_0 - \mathbf{x}^*\| < \delta \text{ and } \|\mathbf{x}(t) - \mathbf{x}^*\| > \varepsilon$$

then $\mathbf{x}^*$ is unstable

Linear dynamical systems; linearising $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where **A** is constant and includes *system characteristics* (mass, damping, stiffness) and (possibly) *force characteristics*.

Solution: $\mathbf{x}(t) = K\mathbf{c}e^{\lambda t}$ => $\dot{\mathbf{x}}(t) = K\lambda\mathbf{c}e^{\lambda t}$

=> $$K\lambda\mathbf{c}e^{\lambda t} = K\mathbf{A}\mathbf{c}e^{\lambda t}$$

Hence, λ is an eigenvalue of the matrix **A**

$$\mathbf{A}\mathbf{c} = \lambda\mathbf{c} \quad => \quad \left|\mathbf{A} - \lambda\mathbf{I}\right| = 0$$

$$\mathbf{x}(t) = \sum K_i \mathbf{c}_i \, e^{\lambda_i t}$$ when is this stable?

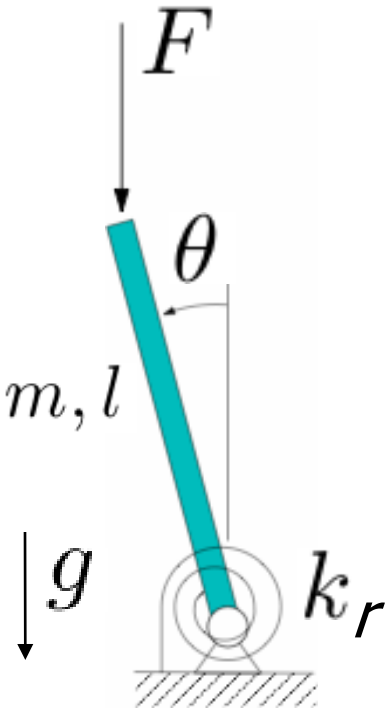
a) When $\lambda_i \in \mathbb{R}$ and $\lambda_i < 0 \quad i = 1, \ldots n$

b) When $\lambda_i \in \mathbb{C}$ and $\mathrm{Re}(\lambda_i) \leq 0 \quad i = 1, \ldots n$

c) Any combination of a) and b)

Non-linear dynamical systems

Linearisation about equilibrium points



Equilibrium points?

Stable at $\theta = 0$?

Satellite system

$$V = -\frac{km}{r}$$