Dynamisc & Stability

AE3-914

Linear dynamical systems; linearising $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where **A** is constant and includes *system characteristics* (mass, damping, stiffness) and (possibly) *force characteristics*.

Solution: $\mathbf{x}(t) = K\mathbf{c}e^{\lambda t}$ => $\dot{\mathbf{x}}(t) = K\lambda\mathbf{c}e^{\lambda t}$

=> $K\lambda\mathbf{c}e^{\lambda t} = K\mathbf{A}\mathbf{c}e^{\lambda t}$

Hence, λ is an eigenvalue of the matrix **A**

$\mathbf{A}\mathbf{c} = \lambda\mathbf{c}$ => $|\mathbf{A} - \lambda\mathbf{I}| = 0$

$$\mathbf{x}(t) = \sum K_i \mathbf{c}_i \, e^{\lambda_i t}$$ when is this stable?

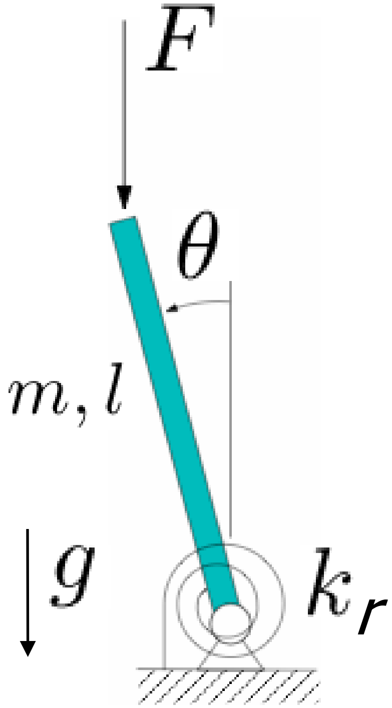
a) When $\lambda_i \in \mathbb{R}$ and $\lambda_i < 0 \quad i = 1, \ldots n$

b) When $\lambda_i \in \mathbb{C}$ and $\mathrm{Re}(\lambda_i) \leq 0 \quad i = 1, \ldots n$

c) Any combination of a) and b)

Non-linear dynamical systems

Linearisation about equilibrium points



Equilibrium points?

Stable at $\theta = 0$?

Stability of conservative systems

Conservative system:

Applied forces can be derived from the potential energy *V*

$$Q = -\frac{\partial V}{\partial q}$$

Equilibrium:

$$\dot{q} = \ddot{q} = 0$$

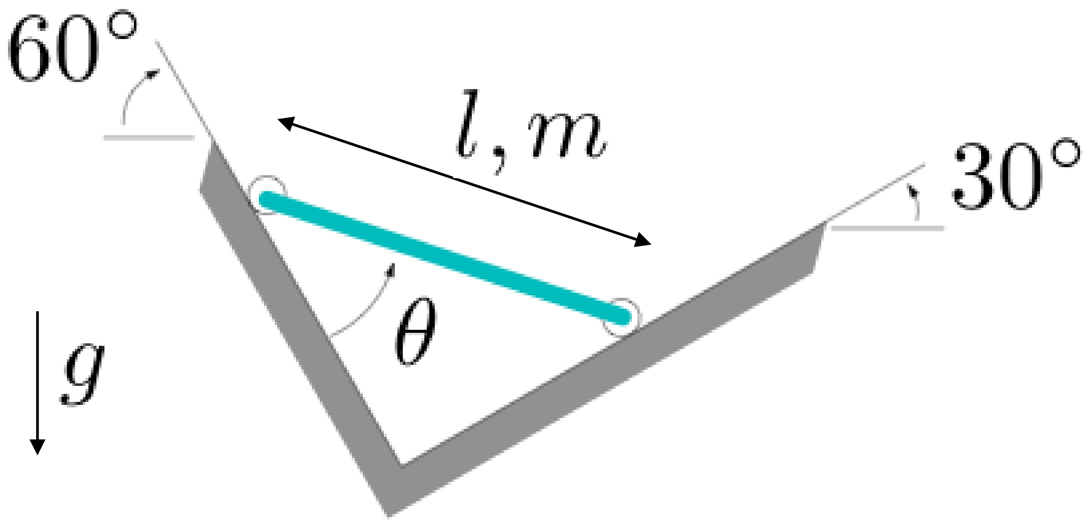
$$T + V = \mathbb{E};\ T = 0 \ \Rightarrow\ V = \mathbb{E} = \text{constant}$$

$$\left.\frac{\partial V}{\partial q}\right|_{q=q^*} = 0 \ \Rightarrow\ q^*$$ is an equilibrium point

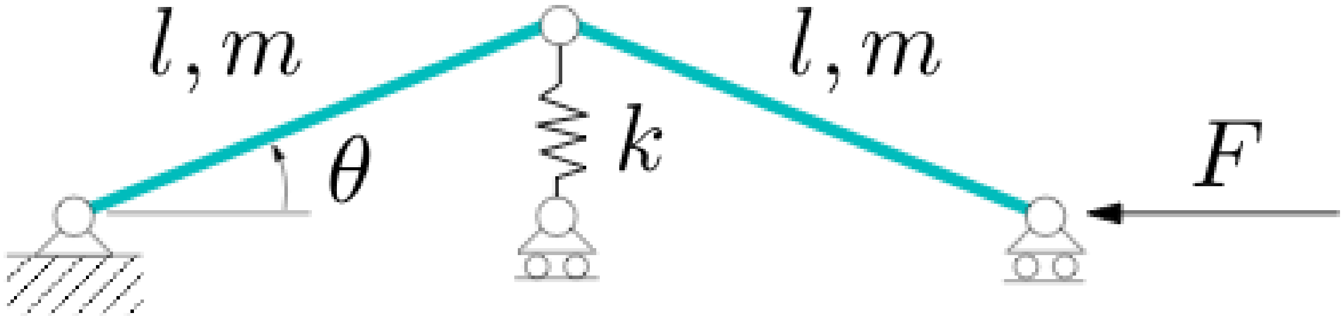
Stability:

$$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q^*} > 0 \;\Rightarrow\; V(q^*) \text{ is a minimum: } \textbf{STABLE}$$

$$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q=q^*} < 0 \;\Rightarrow\; V(q^*) \text{ is a maximum: } \textbf{UNSTABLE}$$



Is the equilibrium position stable?



Natural length of spring is 0

Maximal F for which horizontal position is stable?

Stability of Lagrangian systems

Lagrangian system:

Applied forces can be derived from a generalised potential *V*

Same procedure as for conservative systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Equilibrium:

$$\dot{q} = \ddot{q} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

Equilibrium condition: $-\frac{\partial L}{\partial q} = 0$ with $\dot{q} = 0$

$$L = T - V = T_2 + T_1 + T_0 - V$$

If $\dot{q} = 0$ then $T_2 = T_1 = 0$ and then...

Equilibrium condition: $$\frac{\partial(V - T_0)}{\partial q} = 0$$

$$V_{\text{eff}} = V - T_0$$

is defined as the **effective potential**

Equilibrium configuration

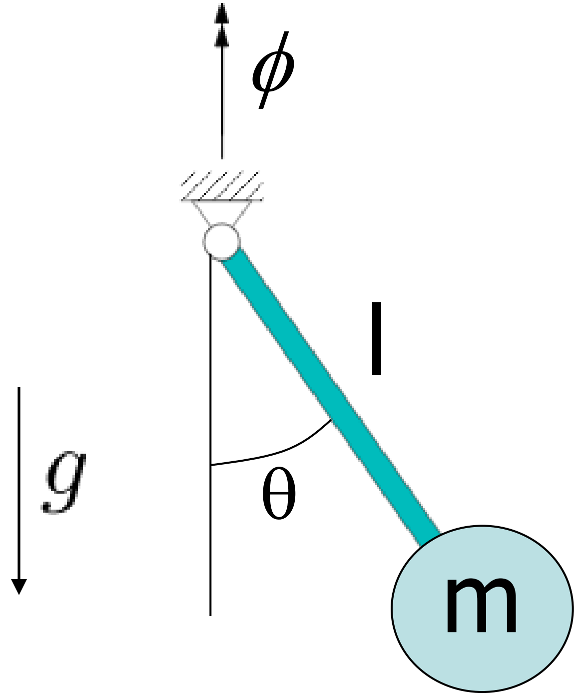
$$\frac{\partial V_{\mathrm{eff}}}{\partial q} = 0$$

V_{eff} is a minimum: **STABLE**

V_{eff} is a maximum: **UNSTABLE**

The effective potential is also present in the Jacobi integral (Török, pg. 124)

$$h = T_2 - T_0 + V = T_2 + V_{\text{eff}}$$



Spherical pendulum

Is the steady motion stable?

Satellite system

$$V = -\frac{km}{r}$$