Dynamics & Stability

AE3-914

Gyrodynamics

$$I_3 = I_s \qquad I_1 = I_2 = I$$

$$T = \frac{1}{2}\left[I(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + I_s(\dot{\phi}\cos\theta + \dot{\psi})^2\right]$$

Steady precession

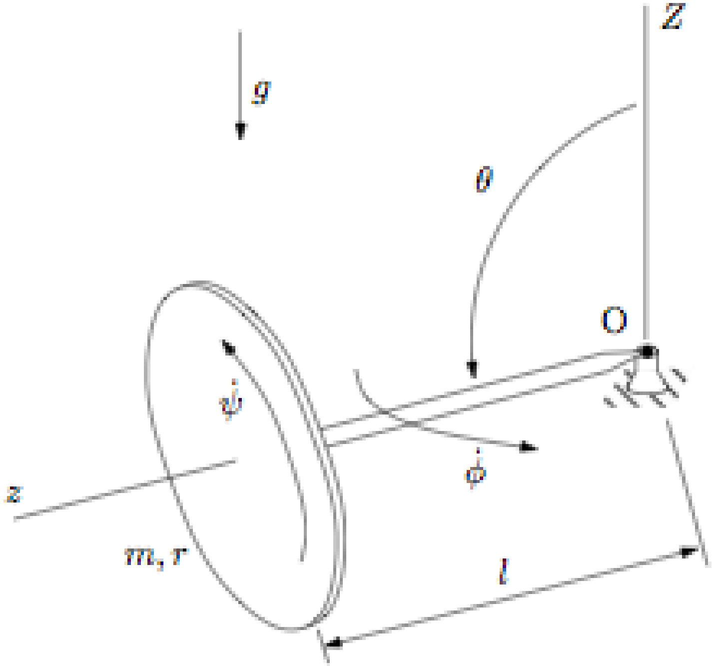
$$\ddot{\theta} = \dot{\theta} = 0 \qquad \frac{\partial R}{\partial \theta} = \frac{dV_{eff}}{d\theta} = 0$$

$$R\left(\theta,\dot{\theta},C_\phi,C_\psi\right)=-\frac{1}{2}I\dot{\theta}^2+\frac{\left(C_\phi-C_\psi\cos\theta\right)^2}{2I\sin^2\theta}+\frac{C_\psi^2}{2I_s}+mgl\cos\theta$$

$$V_{eff}(\theta)=\frac{\left(C_\phi-C_\psi\cos\theta\right)^2}{2I\sin^2\theta}+\frac{C_\psi^2}{2I_s}+mgl\cos\theta$$

Eq. of motion: $$\frac{d}{dt}\left(\frac{\partial R}{\partial\dot{\theta}}\right)-\frac{\partial R}{\partial\theta}=0$$

There is an error in equation (4.38) and on top of page 223 of the textbook corresponding to this course.

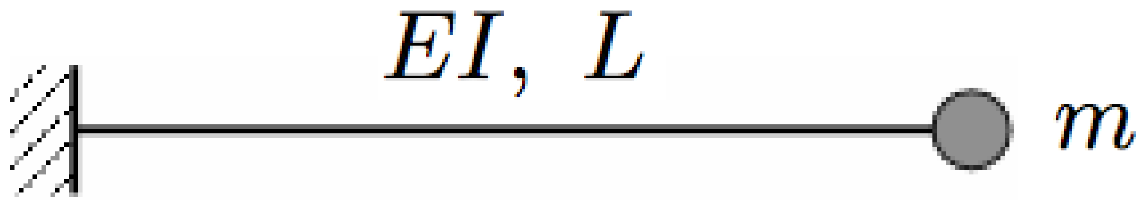


Steady precession for $\theta = 90^\circ$?

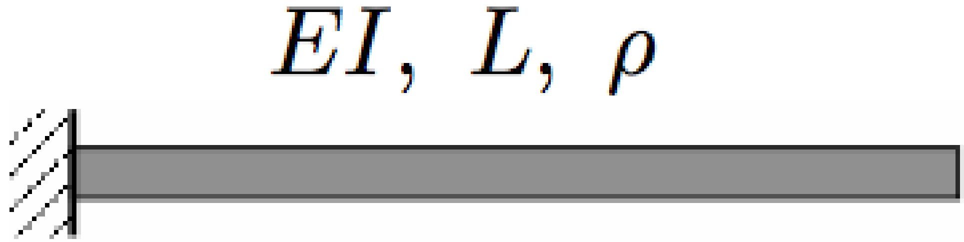
Steady precession for $\theta = 60^\circ$?

This slide corresponds to a sample problem that can be found in the assignments section of this course

Calculus of Variations

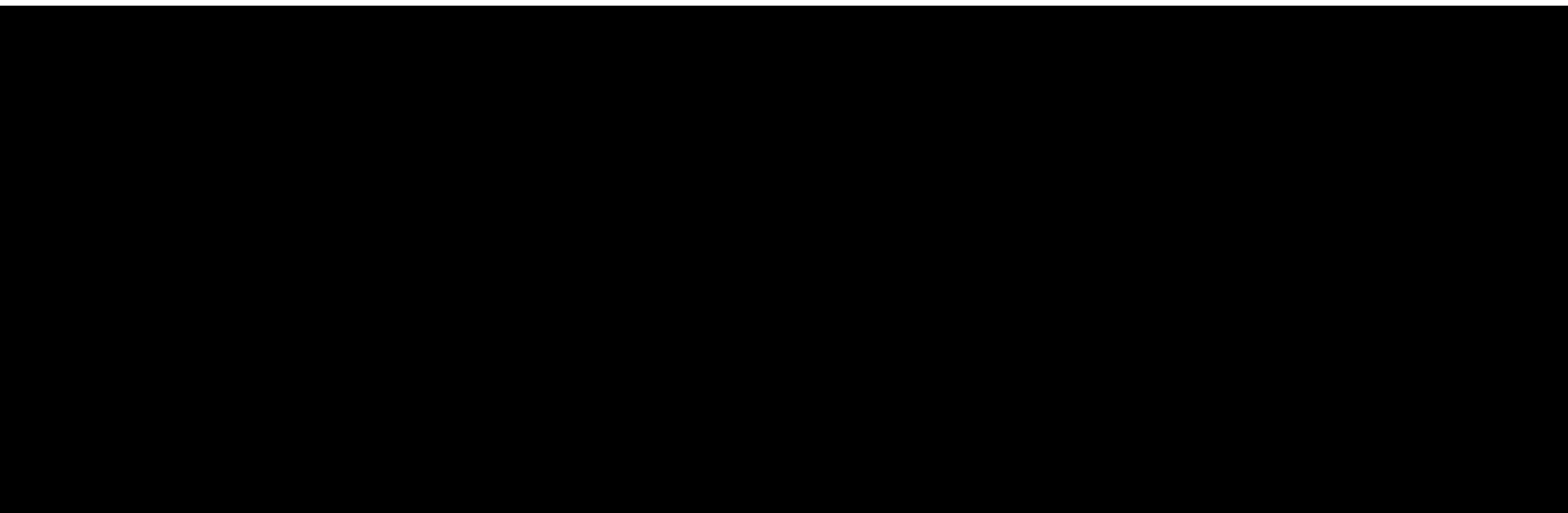


$$m\ddot{u} + \frac{3EI}{L^3}u = 0$$



Equations of motion?

What is the shortest path between two points?



$$ds = \sqrt{1 + [y'(x)]^2}\, dx$$

$$s_{ab}(y) = \int_{x_a}^{x_b} \sqrt{1 + [y'(x)]^2}\, dx$$

= integral functional

Find $y(x)$ such that

$$\int_{x_a}^{x_b} \sqrt{1 + [y'(x)]^2}\, dx$$ is minimal

= variational problem (*calculus of variations*)

What is the fastest path between two points?

$$dt = \frac{ds}{v}$$

$$v = \sqrt{2g\,y(x)}$$

$$t_{ab} = \int_{x_a}^{x_b} \frac{\sqrt{1 + [y'(x)]^2}}{\sqrt{2g\,y(x)}}\,dx$$

Find $y(x)$ such that

$$t_{ab}(y) = \int_{x_a}^{x_b} \frac{\sqrt{1+[y'(x)]^2}}{\sqrt{2g\ y(x)}} dx$$

is minimal

$$s_{ab} = s_{ab}(y) \qquad\qquad t_{ab} = t_{ab}(y)$$

This is what we call **functionals:**
Real-valued expressions of functions
defined within an interval

Calculus of variations

Find the function *y(x)* that minimizes a functional

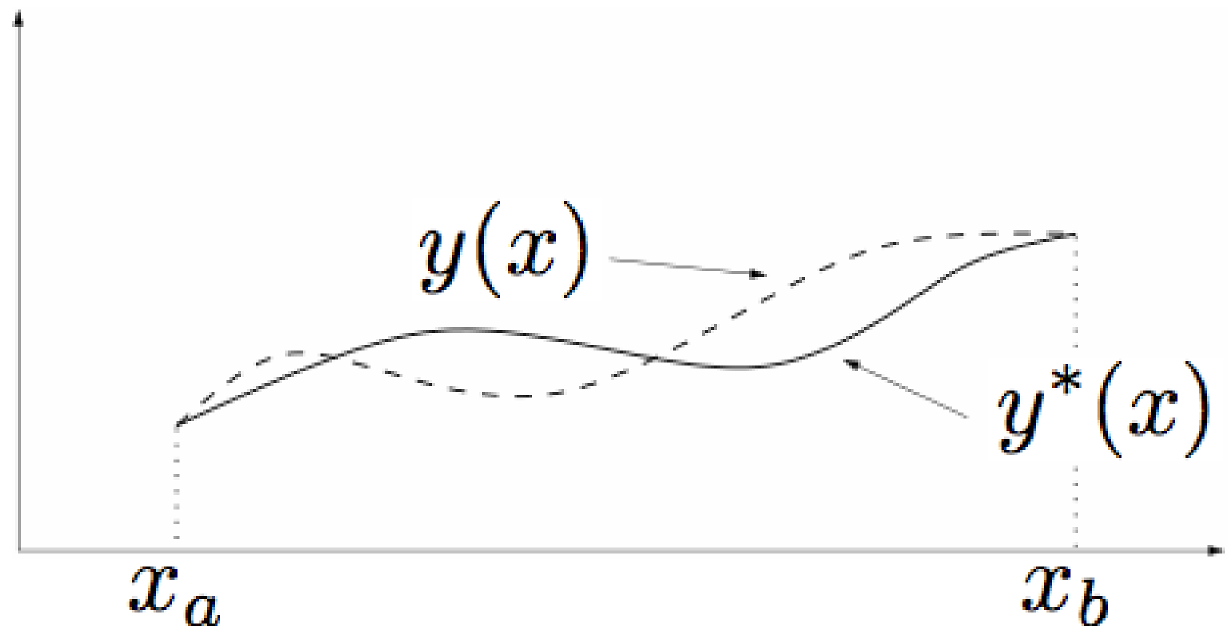
$$I(y) = \int_{x_a}^{x_b} F(x, y, y')\, dx$$

For normal functions one has that if

$$f'(x^*) = 0$$

then x^* provides an extremum (maximum or minimum) to f

For functionals one needs to define what is understood as a variation



If y^* provides an extremal to a functional I,

$$I(y^* + \varepsilon\eta) \geq I(y^*)$$

for any arbitrary perturbation η such that

$$\eta(x_a) = \eta(x_b) = 0$$

$I = I(\varepsilon)$; if I is an extremal at y* then

$$\frac{d}{d\varepsilon} I(y^* + \varepsilon\eta)\Big|_{\varepsilon=0} = 0$$

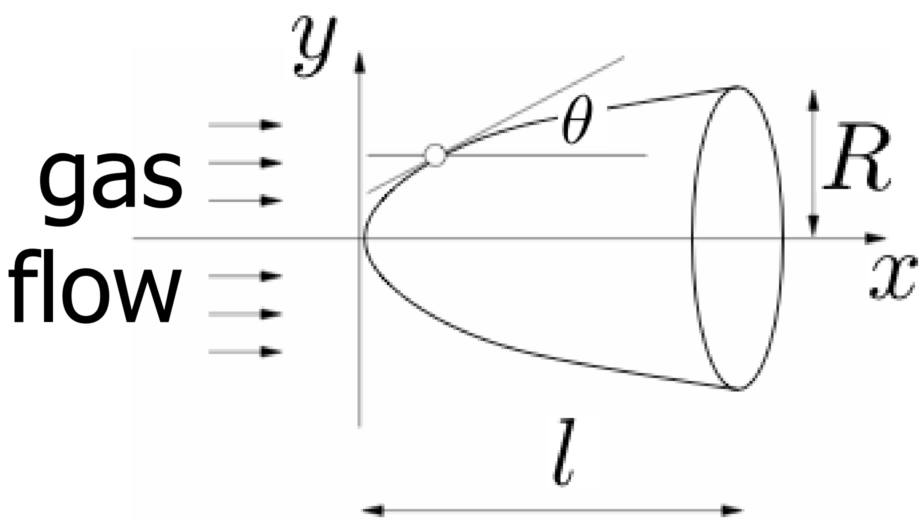
Fundamental lemma:

$$\int_{x_a}^{x_b} G(x)\eta(x)\,dx = 0 \quad \forall \eta \Big| \eta(x_a) = \eta(x_b) = 0$$

implies that $G(x) \equiv 0 \quad \forall x \in [a, b]$

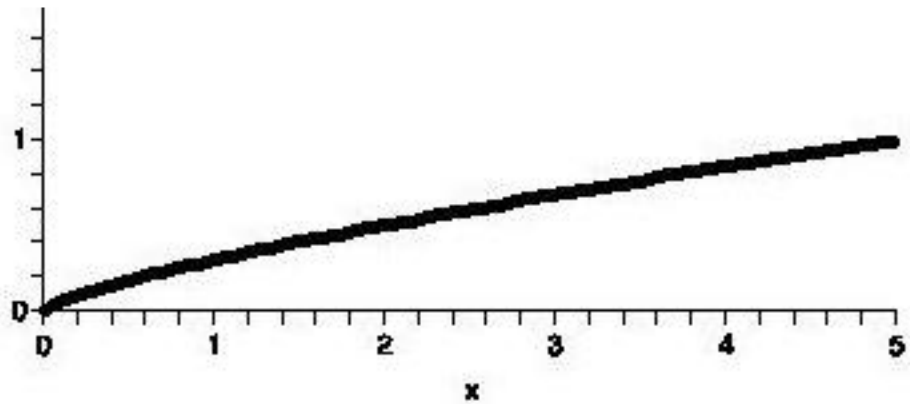
Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$



Shape *y(x)* for minimal resistance?

(assume axial symmetry about *x*)



$$l = 5\,\mathrm{m}$$

$$R = 1\,\mathrm{m}$$

$$y = R\left(\frac{x}{l}\right)^{3/4}$$

Variational operator

$$
\begin{aligned}
y(x) &= y^*(x) + \varepsilon\eta(x) \\
&= y^*(x) + \delta y^*(x)
\end{aligned}
$$

$$I(y + \delta y) = I(y) + \delta I$$

Function *y* provides an extremal of *I* if

$$\delta I = 0$$

"δ" is comparable to a differential

$$
\begin{aligned}
\delta I &= \int_{x_a}^{x_b} \delta F(x, y, y')\, dx \\
&= \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right)\, dx
\end{aligned}
$$

$$= \int_{x_a}^{x_b} \left(\frac{\partial F}{\partial y} \delta y - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \delta y \right) \, dx + \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_a}^{x_b}$$

Since δy represents any variation of y

and $\delta y(x_a) = \delta y(x_b) = 0$ we end again with

Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Natural boundary conditions

$$\delta I = \int_{x_a}^{x_b} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \delta y \, dx + \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_a}^{x_b}$$

where $y(x_a) = y_a$

but nothing is said about $y(x_b)$

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_b} = 0$$

is a natural boundary condition